# The Schur algorithm for generalized Schur functions II

Alpay, D.; Azizov, T.Ya.; Dijksma, A.; Langer, H.

*Published in:*
Monatshefte fur mathematatik

*DOI:*
10.1007/S00605-002-0528-6

Monatsh. Math. 138, 1–29 (2003)
DOI 10.1007/s00605-002-0528-6

Monatshefte für
Mathematik
Printed in Austria

# The Schur Algorithm for Generalized Schur Functions II: Jordan Chains and Transformations of Characteristic Functions

By

**D. Alpay**[1], **T. Ya. Azizov**[2], **A. Dijksma**[3], and **H. Langer**[4]

[1] Ben-Gurion University of the Negev, Beer Sheva, Israel
[2] Voronezh State University, Voronezh, Russia
[3] University of Groningen, The Netherlands
[4] Vienna University of Technology, Austria

Received October 31, 2001; in revised form August 21, 2002
Published online November 15, 2002 

Dedicated to Professor Edmund Hlawka on the occasion of his 85th birthday

**Abstract.** In the first paper of this series (Daniel Alpay, Tomas Azizov, Aad Dijksma, and Heinz Langer: The Schur algorithm for generalized Schur functions I: coisometric realizations, Operator Theory: Advances and Applications 129 (2001), pp. 1–36) it was shown that for a generalized Schur function $s(z)$, which is the characteristic function of a coisometric colligation $V$ with state space being a Pontryagin space, the Schur transformation corresponds to a finite-dimensional reduction of the state space, and a finite-dimensional perturbation and compression of its main operator. In the present paper we show that these formulas can be explained using simple relations between $V$ and the colligation of the reciprocal $s(z)^{-1}$ of the characteristic function $s(z)$ and general factorization results for characteristic functions.

2000 Mathematics Subject Classification: 46C20, 47A48, 30D99
Key words: Schur algorithm, generalized Schur function, operator colligation, Schur determinant, Pontryagin space

## 1. Introduction

By $S_0$ we denote the class of all *Schur functions*: these are the functions $s$ defined and holomorphic on the open unit disc $\mathbb{D}$ and such that $|s(z)| \leqslant 1, z \in \mathbb{D}$. If $s$ is not a constant of modulus 1 then the Schur transformation (see [14], [15])

$$s_1(z) = \frac{1}{z} \frac{s(z) - s(0)}{1 - s(z)s(0)^*} \tag{1.1}$$

is defined and belongs again to the class $S_0$. The larger class of generalized Schur functions was introduced in [13], [9], see also [5], [4]: It consists of the functions $s$

The research for this paper was supported by grants from the Netherlands Organization for Scientific Research NWO 047-008-008 and NWO 61-453 and from the Russian Foundation for Basic Research RFBR 99-01-00391. A. Dijksma and H. Langer acknowledge support through Harry T. Dozor fellowships at the Ben-Gurion University of the Negev, Beer-Sheva, Israel, in the years 1999 and 2000, respectively. We thank the referee for his/her remarks.

which are meromorphic in $\mathbb{D}$ and such that the kernel

$$S_s(z,\zeta) := \frac{1 - s(z)s(\zeta)^*}{1 - z\zeta^*},$$

which is defined for $z, \zeta \in \mathscr{D}(s)$, the domain of holomorphy of $s$, has a finite number of negative squares; if this number is $\kappa$ the function $s$ belongs by definition to the class $S_\kappa$. It turns out (see [5]) that $s \in S_\kappa$ if and only if $s$ is meromorphic in $\mathbb{D}$ with $\kappa$ poles, counted according to their multiplicities, and such that

$$\limsup_{|z|\uparrow 1} |s(z)| \leqslant 1.$$

Further, by $A^0$ we denote the set of functions which are defined and holomorphic in a neighbourhood of $z = 0$, and we set $S_\kappa^0 := S_\kappa \cap A^0$, $S^0 := \cup_{\kappa=0}^\infty S_\kappa^0$. Evidently, $S_0^0 = S_0$.

For a function $s \in A^0$ the Schur transformation has been generalized in [5] and [10] as follows: If $|s(0)| < 1$ the Schur transformation $s_1$ is still defined by (1.1), if $|s(0)| > 1$ it is defined by the formula

$$s_k(z) = z^k \frac{1 - s(z)s(0)^*}{s(z) - s(0)}, \tag{1.2}$$

where $k\ (>0)$ is the order of the pole at zero of the fraction on the right hand side of (1.2), and if $|s(0)| = 1$ then the Schur transformation is defined by

$$s_{2k+q}(z) = z^q \frac{(Q(z) - z^k)s(z) - s(0)Q(z)}{s(0)^*Q(z)s(z) - (Q(z) + z^k)}; \tag{1.3}$$

here $k$ is given by the Taylor expansion of $s(z)$ at $z = 0$ as the index of the first non-vanishing coefficient $\sigma_k \neq 0$, $k \geqslant 1$, that is,

$$s(z) = \sigma_0 + \sigma_k z^k + \sigma_{k+1} z^{k+1} + \cdots,$$

and

$$Q(z) = c_0 + \cdots + c_{k-1} z^{k-1} - (c_{k-1}^* z^{k+1} + \cdots + c_0^* z^{2k}),$$

where the coefficients $c_j$, $j = 0, 1, \ldots, c_{k-1}$, are defined by the relation

$$(s(z) - \sigma_0)(c_0 + c_1 z + \cdots + c_n z^n + \cdots) \equiv \sigma_0 z^k;$$

finally, $q\ (\geqslant 0)$ is the order of the pole at zero of the fraction on the right hand side of (1.3). In (1.2) and (1.3), respectively, the index $k$ or $2k + q$ is used because of the fact that, for example, for a rational function $s$ the degree of the transformed function is the degree of $s$ minus $k$ or minus $2k + q$, respectively, whereas for the formula (1.1) the reduction of the degree in passing from $s$ to $s_1$ is just one.

To a function $s \in A^0$ there corresponds a minimal coisometric colligation $V$ in a space $\mathscr{H} \oplus \mathbb{C}$ such that $s$ is the characteristic function of $V$, see [8], [4] and also Section 2 below. If the coisometric colligation is

$$V = \begin{pmatrix} T & u \\ \langle \cdot, v \rangle & \gamma \end{pmatrix} : \begin{pmatrix} \mathscr{H} \\ \mathbb{C} \end{pmatrix} \to \begin{pmatrix} \mathscr{H} \\ \mathbb{C} \end{pmatrix}$$

with elements $u, v$ of the Krein space $\mathscr{H}$, a bounded linear operator $T$ in $\mathscr{H}$ and a number $\gamma \in \mathbb{C}$, its characteristic function is by definition

$$s(z) = \gamma + z\langle (I - zT)^{-1}u, v\rangle.$$

If, in particular, $s \in S_\kappa^0$ the Krein space $\mathscr{H}$ is in fact a Pontryagin space with negative index $\kappa$.

For $s \in A^0$ with corresponding minimal coisometric colligation $V$ we consider its Schur transformation given by one of the formulas (1.1), (1.2), (1.3) above, and the corresponding minimal coisometric colligation $V_1$, $V_k$, or $V_{2k+q}$, which for a moment we denote by $\tilde{V}$. In [2] the entries of the colligation $\tilde{V}$ were expressed by the entries of $V$. It was shown that the Krein space $\tilde{\mathscr{H}}$ is a subspace of $\mathscr{H}$ which is in case (1.1) of codimension one, in case (1.2) of codimension $k$, and in case (1.3) of codimension $2k + q$, and that $\tilde{T}$ is a compression of a finite dimensional perturbation of $T$ to this subspace. The corresponding formulas, which yield that the Schur transformation $s_1$, $s_k$ or $s_{2k+q}$ is the characteristic function of $\tilde{V}$, are derived in [2] by straightforward calculations.

In the present paper we consider some operators which are related to the colligation $V$, and their Jordan chains at the eigenvalue zero more closely. The reason for this is that the formulas (1.1), (1.2), (1.3) can all be reduced to fractional linear transformations and multiplications by $z^{-1}$ or a power of $z$: for (1.1) and (1.2) this is evident, for (1.3) this follows if we write $s_{2k+q}(z)$ as

$$s_{2k+q}(z) = \sigma_0 z^q \left(1 + \frac{z^k}{\frac{z^k\sigma_0}{s(z)-\sigma_0} - Q(z)}\right).$$

On the other hand, for an operator colligation $V$ with characteristic function $s$, a fractional linear transformation $\hat{s}$ of $s$ is in general the characteristic function of a colligation $\hat{V}$ where the main operator $\hat{T}$ is a one-dimensional perturbation of the main operator $T$ of $V$, and if zero is an eigenvalue of $\hat{T}$ with a Jordan chain of length $k$ then $\hat{s}$ has a zero at $z = 0$ of order $k$. We use these facts in the present paper in order to derive the colligations corresponding to the Schur transformation in (1.1), (1.2), and (1.3). Of course, the formulas are the same as in [2], however, in our opinion, the proofs given here lead to a better insight into the structure of the colligations.

Section 2 contains some simple facts about one-dimensional perturbations of an operator $S$, in particular, the colligation of the inverse of a characteristic function is described. Together with a knowledge of the Jordan chains at the eigenvalue zero of a one-dimensional perturbation of $T$, the colligations corresponding to the formulas (1.1) and (1.2) can easily be derived, see Section 3. The case of formula (1.3) is more complicated and is split into several steps in Section 4. For the fact that the obtained colligation is coisometric and minimal we refer to the calculations of [2].

Now we start from a function $s = s_0 \in S_\kappa^0$ which is not a constant of modulus one, and consider the Schur transformation $s_1$, $s_k$ or $s_{2k+q}$ according to one of the formulas (1.1), (1.2), or (1.3). If this new function is not a constant of modulus one, its Schur transformation can be considered, and this procedure can be continued as long as the outcome is not a constant of modulus one. Thus we obtain a finite or infinite sequence of functions $s_{j_n}$, $j_0 = 0$, $j_n \in \{1, 2, \ldots\}$ for $n = 1, 2, \ldots, n_0$ or $n = 1, 2, \ldots$, such that $s_{j_n}$ belongs to a class $S^0_{\kappa_n}$, where $\kappa_0 = \kappa$. The sequence of the $\kappa_n$ is nonincreasing and it turns out that, beginning from a certain index, it consists only of zeros, that is, the functions $s_{j_n}$ are finally Schur functions. In Section 5 we show that for the function $s \in S^0_\kappa$ the index $\kappa$ is determined by the number of sign changes in the sequence of the Schur determinants of $s$, and also relations between this sequence of Schur determinants and the sequence $(j_n)$ are established.

The characteristic function of an operator colligation is, by definition, always holomorphic at zero. For this reason in this note the function $s$ and its Schur transformation are restricted to the class $S^0$, and in the formula (1.2) the number $k$ may be $> 1$, and in (1.3) the number $q$ may be $> 0$. In fact, they have to be chosen as the smallest integers such that the Schur transformation is holomorphic at zero. In another paper we shall study the Schur transformation using reproducing kernel Pontryagin spaces of the functions of class $S_\kappa$. Then the function $s$ and also the transformed functions can have a pole at zero.

Finally we mention that in the case $\kappa = 0$ according to [11] the reduction process of the coisometric colligation as explained in [2] and in the present paper, which is implied by consecutive application of the Schur transformation, is equivalent to Andersson's algorithm for the inverse problem for certain Sturm–Liouville equations, see [1].

This paper is dedicated to Professor Edmund Hlawka. Born at the time when Schur discovered his famous transformation, he has influenced mathematics and generations of mathematicians during the last century in a very essential way.

## 2. Coisometric Colligations

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Krein space. With elements $u, v \in \mathcal{H}$, a bounded linear operator $T$ in $\mathcal{H}$ and a number $\gamma \in \mathbb{C}$ we consider the following operator $V$ in $\mathcal{H} \oplus \mathbb{C}$:

$$V = \begin{pmatrix} T & u \\ \langle \cdot, v \rangle & \gamma \end{pmatrix}. \tag{2.1}$$

Often $V$ is called a *colligation* and $T$ its *main operator*. The colligation $V$ is *coisometric* if $VV^* = I$, where $I$ stands for the identity operator on any space, in this case for the identity operator in $\mathcal{H} \oplus \mathbb{C}$. Evidently, the coisometry of the colligation (2.1) is equivalent to the relations

$$TT^* + \langle \cdot, u \rangle u = I, \quad Tv + \gamma^* u = 0, \quad \langle v, v \rangle + |\gamma|^2 = 1. \tag{2.2}$$

If follows from the first relation that the adjoint $T^*$ of the main operator $T$ of the coisometric colligation $V$ is a contraction in $\mathcal{H}$:

$$\langle T^* x, T^* x \rangle = \langle x, x \rangle - |\langle x, u \rangle|^2 \leqslant \langle x, x \rangle, \quad x \in \mathcal{H}.$$

The *characteristic function* $s_V(z)$ of the colligation $V$ is

$$s_V(z) := \gamma + z\langle (I - zT)^{-1} u, v \rangle. \tag{2.3}$$

It is defined and holomorphic at least in a neighbourhood of $z=0$, and if we write

$$s_V(z)=\sum_{j=0}^{\infty}\sigma_j z^j \tag{2.4}$$

then we have $\sigma_0=\gamma$ and $\sigma_j=\langle T^{j-1}u,v\rangle$, $j=1,2,\ldots$. The relations (2.2) imply for $i,j=0,1,\ldots$

$$\langle T^{*j}v,T^{*i}v\rangle=\begin{cases}1-|\sigma_0|^2-|\sigma_1|^2-\cdots-|\sigma_i|^2, & i=j,\\ -(\sigma_0^*\sigma_{i-j}+\sigma_1^*\sigma_{i-j+1}+\cdots+\sigma_j^*\sigma_i), & i>j.\end{cases} \tag{2.5}$$

For the simple proof of this formula we refer to [2, (4.7)].

The space $\mathscr{H}$, if equipped with the inner product $\langle x,y\rangle_-:=-\langle x,y\rangle$, $x,y\in\mathscr{H}$, is denoted by $\mathscr{H}_-$. We need the following well-known lemma. Its proof is straightforward, see also [6], [2].

**Lemma 2.1.** *If*

$$U=\begin{pmatrix} S & f\\ \langle\cdot,g\rangle & \gamma\end{pmatrix} \tag{2.6}$$

*is a coisometric colligation in $\mathscr{H}\oplus\mathbb{C}$ with $\gamma\neq 0$, and $s_U(z)$ denotes its characteristic function, then the colligation*

$$\widehat{U}:=\begin{pmatrix} S-\dfrac{\langle\cdot,g\rangle f}{\gamma} & \pm\dfrac{f}{\gamma}\\ \mp\dfrac{\langle\cdot,g\rangle}{\gamma} & \dfrac{1}{\gamma}\end{pmatrix}$$

*is coisometric in $\mathscr{H}_-\oplus\mathbb{C}$ and its characteristic function is given by $s_{\widehat{U}(z)=\frac{1}{s_U(z)}}$. Moreover, if the coisometric colligation $U$ is minimal, then so is $\widehat{U}$.*

*Remark 2.2.* The system theoretic interpretation of Lemma 2.1 is that the input $\alpha$ and the output $\beta$ for the realization $U$ of $s(z)=s_U(z)$,

$$U\begin{pmatrix}h\\ \alpha\end{pmatrix}=\begin{pmatrix}k\\ \beta\end{pmatrix},\qquad h,k\in\mathscr{H},\qquad \alpha,\beta\in\mathbb{C}, \tag{2.7}$$

change roles in the realization $\widehat{U}$ of $\frac{1}{s(z)}$: the relation (2.7) is equivalent to

$$\widehat{U}\begin{pmatrix}h\\ \pm\beta\end{pmatrix}=\begin{pmatrix}k\\ \pm\alpha\end{pmatrix}.$$

We do not know of a similar interpretation for the realizations that appear elsewhere in the paper, for example in Lemma 4.1.

**Lemma 2.3.** *If $S$ is a bounded linear operator in an inner product space $(\mathscr{H},\langle\cdot,\cdot\rangle)$, $f,g\in\mathscr{H}$, the integer $q\in\{1,2,\ldots\}$ is defined by the relations*

$$\langle f,g\rangle=\langle Sf,g\rangle=\cdots=\langle S^{q-2}f,g\rangle=0,\quad \langle S^{q-1}f,g\rangle\neq 0, \tag{2.8}$$

*and $\tilde{S}:=S-\langle\cdot,g\rangle f$, then*

$$\frac{1}{z^{q-1}}\frac{\langle (I-zS)^{-1}f,g\rangle}{1+z\langle (I-zS)^{-1}f,g\rangle}=\langle f,\tilde{S}^{*(q-1)}g\rangle+z\langle (I-z\tilde{S})^{-1}f,\tilde{S}^{*q}g\rangle. \tag{2.9}$$

*Proof.* From the definition we obtain

$$\langle (I - z\tilde{S})^{-1} f, g\rangle = \frac{\langle (I - zS)^{-1} f, g\rangle}{1 + z\langle (I - zS)^{-1} f, g\rangle}. \tag{2.10}$$

The assumptions (2.8) imply the same for $\tilde{S}$:

$$\langle f, g\rangle = \langle \tilde{S} f, g\rangle = \cdots = \langle \tilde{S}^{q-2} f, g\rangle = 0, \quad \langle \tilde{S}^{q-1} f, g\rangle \neq 0.$$

These relations allow to write

$$\begin{aligned}\langle (I - z\tilde{S})^{-1} f, g\rangle &= \sum_{j=0}^{\infty} z^j \langle \tilde{S}^j f, g\rangle = \sum_{j=q-1}^{\infty} z^j \langle \tilde{S}^j f, g\rangle \\ &= z^{q-1}\left\langle \left(\tilde{S}^{q-1} + z\tilde{S}^q \sum_{j=0}^{\infty} (z\tilde{S})^j\right) f, g\right\rangle \\ &= z^{q-1}(\langle f, \tilde{S}^{*(q-1)} g\rangle + z\langle (I - z\tilde{S})^{-1} f, \tilde{S}^{*q} g\rangle).\end{aligned}$$

Combining this formula with formula (2.10) we obtain formula (2.9). □

*Remark 2.4.* With the (not necessarily coisometric) colligation $U$ from (2.6) and its characteristic function $s_U$ the statement of the lemma can be formulated as follows: The function

$$\tilde{s}(z) := \frac{1}{z^q} \frac{s_U(z) - \gamma}{1 + (s_U(z) - \gamma)}$$

is the characteristic function of the colligation

$$\tilde{U} := \begin{pmatrix} \tilde{S} & f \\ \langle \cdot, \tilde{S}^{*q} g\rangle & \langle f, \tilde{S}^{*(q-1)} g\rangle \end{pmatrix}.$$

## 3. The Case $|\sigma_0| \neq 1$

**3.1.** Suppose now that $|\gamma| \neq 1$. With the colligation (2.1) we associate the operator

$$\hat{T} := T + \frac{\gamma^*}{1 - |\gamma|^2} \langle \cdot, v\rangle u \tag{3.1}$$

and the characteristic function $s_V(z)$ as in (2.3), (2.4). Further, let $k$ be the integer which is determined by the relation

$$k := \min\{l : l \geqslant 1, \sigma_l \neq 0\}, \tag{3.2}$$

that is, $\sigma_k \neq 0$ and, if $k > 1$, then

$$\sigma_1 = \cdots = \sigma_{k-1} = 0. \tag{3.3}$$

**Lemma 3.1.** *If the colligation $V$ in* (2.1) *is coisometric and has the property $|\gamma| \neq 1$, then $\hat{T} v = 0$. If, additionally, for its characteristic function $s_V$ the integer $k$ in* (3.2) *is $> 1$, then the elements $v, T^* v, T^{*2} v, \ldots, T^{*(k-1)} v$ form a Jordan chain of $\hat{T}$ at the eigenvalue zero*:

$$\hat{T} v = 0, \quad \hat{T} T^* v = v, \ \ldots, \ \hat{T} T^{*(k-1)} v = T^{*(k-2)} v.$$

*If $V$ is minimal then for each $l \in \{1, 2, \dots, k\}$,*

$$\ker \hat{T}^l = \operatorname{span}\{v, T^*v, \dots, T^{*(l-1)}v\}.$$

*Proof.* The relations (2.2) imply

$$\hat{T}v = Tv + \frac{\gamma^*}{1-|\gamma|^2}\langle v, v\rangle u = Tv + \gamma^* u = 0,$$

and, similarly, for $1 \leqslant j \leqslant k-1$,

$$\begin{aligned}\hat{T}T^{*j}v &= TT^{*j}v + \frac{\gamma^*}{1-|\gamma|^2}\langle v, T^j v\rangle u \\ &= T^{*(j-1)}v - \langle T^{*(j-1)}v, u\rangle u - \frac{|\gamma|^2}{1-|\gamma|^2}\langle v, T^{j-1}u\rangle u \\ &= T^{*(j-1)}v - \frac{1}{1-|\gamma|^2}\sigma_j^* u = T^{*(j-1)}v.\end{aligned}$$

We prove the last equality of the lemma. The first part of the lemma implies

$$\operatorname{span}\{v, T^*v, \dots, T^{*(l-1)}v\} \subset \ker \hat{T}^l. \tag{3.4}$$

To prove that equality prevails when $V$ is minimal we use a dimension argument. Note that by (2.5) and (3.2) the space on the left has dimension $l$: The Gram matrix $(\langle T^{*j}v, T^{*i}v\rangle)_{i,j=0}^{l-1}$ is equal to $(1-|\gamma|^2)$ times the $l \times l$ identity matrix. The minimality implies that $x \in \ker \hat{T}^l$ if and only if $\langle x, \hat{T}^{*l}T^{*j}v\rangle = 0$, $j = 0, 1, 2, \dots$. By computing $\hat{T}^{*l}T^{*j}v$ for $j = 0, 1, 2, \dots$, we find that

$$\ker \hat{T}^l = \mathscr{M}^\perp, \tag{3.5}$$

where $\mathscr{M} = \{T^{*(l+j)}v + p_j(T^*)v \mid j = 0, 1, 2, \dots\}$, and each $p_j(z)$ is some polynomial in $z$ of degree at most $l-1$. Evidently,

$$\overline{\operatorname{span}}\,\mathscr{M} + \operatorname{span}\{v, T^*v, \dots, T^{*(l-1)}v\} = \mathscr{H}.$$

This decomposition implies $\dim \mathscr{M}^\perp \leqslant l$ and the inclusion (3.4) yields $\ker \hat{T}^l \geqslant l$. Because of (3.5) these inequalities can be replaced by equalities. Consequently, equality prevails in the inclusion (3.4). □

Now we consider a coisometric colligation with the operator $\hat{T}$ from (3.1) as the main operator. Here we have to distinguish between the cases $|\gamma| < 1$ and $|\gamma| > 1$. Set

$$\hat{u} = \frac{u}{\sqrt{|1-|\gamma|^2|}}, \quad \hat{v} = \frac{v}{\sqrt{|1-|\gamma|^2|}}.$$

**Lemma 3.2.** *If $|\gamma| < 1$ then the colligation*

$$\hat{V} := \begin{pmatrix} \hat{T} & \hat{u} \\ \langle \cdot, \hat{v}\rangle & 0 \end{pmatrix}$$

*is coisometric in the space* $\mathscr{H} \oplus \mathbb{C}$; *if* $|\gamma| > 1$ *then the colligation*

$$\hat{V}_- := \begin{pmatrix} \hat{T} & \hat{u} \\ \langle \cdot, \hat{v} \rangle_- & 0 \end{pmatrix}$$

*is coisometric in the space* $\mathscr{H}_- \oplus \mathbb{C}$. *In both cases the characteristic function of the new coisometric colligation* $\hat{V}$ *or* $\hat{V}_-$ *is*

$$\frac{s_V(z) - \gamma}{1 - \gamma^* s_V(z)}.$$

*If* $V$ *is minimal, then* $\hat{V}$ *if* $|\gamma| < 1$ *or* $\hat{V}_-$ *if* $|\gamma| > 1$ *is also minimal.*

*Proof.* Suppose that $|\gamma| < 1$. The coisometry relation $\hat{V}\hat{V}^* = I$ is equivalent to the following identities; here we use the relations (2.2):

$$\begin{aligned} \hat{T}\hat{T}^* + \langle \cdot, \hat{u} \rangle \hat{u} &= \left( T + \frac{\gamma^*}{1 - |\gamma|^2} \langle \cdot, v \rangle u \right) \left( T^* + \frac{\gamma}{1 - |\gamma|^2} \langle \cdot, u \rangle v \right) \\ &\quad + \frac{1}{1 - |\gamma|^2} \langle \cdot, u \rangle u, \\ &= TT^* + \frac{\gamma^*}{1 - |\gamma|^2} \langle \cdot, Tv \rangle u + \frac{\gamma}{1 - |\gamma|^2} \langle \cdot, u \rangle Tv \\ &\quad + \frac{|\gamma|^2}{(1 - |\gamma|^2)^2} \langle v, v \rangle \langle \cdot, u \rangle u + \frac{1}{1 - |\gamma|^2} \langle \cdot, u \rangle u \\ &= TT^* + \langle \cdot, u \rangle u = I, \\ \hat{T}\hat{v} &= \frac{1}{\sqrt{1 - |\gamma|^2}} \left( Tv + \frac{\gamma^*}{1 - |\gamma|^2} \langle v, v \rangle u \right) = 0, \\ \langle \hat{v}, \hat{v} \rangle &= \frac{\langle v, v \rangle}{1 - |\gamma|^2} = 1. \end{aligned}$$

In order to find the characteristic function $s_{\hat{V}}(z)$ we consider the element $\hat{x} = (I - z\hat{T})^{-1}\hat{u}$. It follows that

$$\begin{aligned} \hat{u} = (I - z\hat{T})\hat{x} = \hat{x} - zT\hat{x} - \frac{z\gamma^*}{1 - |\gamma|^2} \langle \hat{x}, v \rangle u, \\ \frac{(I - zT)^{-1}u}{\sqrt{1 - |\gamma|^2}} = \hat{x} - \frac{z\gamma^*}{1 - |\gamma|^2} \langle \hat{x}, v \rangle (I - zT)^{-1} u, \\ \frac{\langle (I - zT)^{-1}u, v \rangle}{\sqrt{1 - |\gamma|^2}} = \langle \hat{x}, v \rangle - \frac{z\gamma^*}{1 - |\gamma|^2} \langle \hat{x}, v \rangle \langle (I - zT)^{-1}u, v \rangle, \\ \langle \hat{x}, v \rangle = \frac{\langle (I - zT)^{-1}u, v \rangle}{\sqrt{1 - |\gamma|^2} \left( 1 - \frac{z\gamma^*}{1 - |\gamma|^2} \langle (I - zT)^{-1}u, v \rangle \right)}, \end{aligned}$$

and we obtain

$$\begin{aligned} s_{\hat{V}}(z) &= z\langle (I - z\hat{T})^{-1}\hat{u}, \hat{v}\rangle = z\frac{\langle \hat{x}, v\rangle}{\sqrt{1-|\gamma|^2}} \\ &= \frac{z\langle (I-zT)^{-1}u, v\rangle}{(1-|\gamma|^2)\left(1 - \frac{z\gamma^*}{1-|\gamma|^2}\langle (I-zT)^{-1}u, v\rangle\right)} \\ &= \frac{s_V(z) - \gamma}{1-\gamma^* s_V(z)}. \end{aligned}$$

The equality

$$\operatorname{span}\{\hat{v}, \hat{T}^*\hat{v}, \dots\} = \operatorname{span}\{v, T^* v, \dots\}.$$

implies that if $V$ is minimal, then $\hat{V}$ is minimal also. In the case $|\gamma| > 1$ the proof is similar. □

**3.2.** Now we consider a function $s \in S_\kappa^0$, $s(z) = \sum_{j=0}^\infty \sigma_j z^j$, with $|\sigma_0| \neq 1$. By $k(\geqslant 1)$ we denote the integer given by (3.2) and by

$$V = \begin{pmatrix} T & u \\ \langle \cdot, v\rangle & \gamma \end{pmatrix} \tag{3.6}$$

a minimal coisometric colligation in the space $\mathscr{H} \oplus \mathbb{C}$ having $s$ as its characteristic function, $\mathscr{H}$ being a Pontryagin space with $\kappa$ negative squares, and $\gamma = \sigma_0$. The assumption $|\sigma_0| \neq 1$ and (2.5) imply that for $l \in \{1, 2, \dots, k\}$ the subspace

$$\operatorname{span}\{v, T^* v, \dots, T^{*(l-1)}v\}$$

is nondegenerate, in fact it is positive if $|\sigma_0| < 1$ and negative if $|\sigma_0| > 1$. (We used this fact already in the proof of Lemma 3.1.) In the second case the inequality $|\sigma_0| > 1$ implies that $k \leqslant \kappa$.

For $l \in \{1, 2, \dots, k\}$, denote by $\mathscr{H}_l$ the subspace

$$\mathscr{H}_l := \{v, T^* v, \dots, T^{*(l-1)}v\}^\perp$$

of $\mathscr{H}$ and by $P_l$ the orthogonal projection onto $\mathscr{H}_l$.

**Theorem 3.3.** *Given the function $s \in S_\kappa^0$ with $|\sigma_0| < 1$ and a corresponding minimal coisometric colligation $V$ in $\mathscr{H} \oplus \mathbb{C}$ as in* (3.6) *with $\gamma = \sigma_0$. Let $k$ be given by* (3.2), *let $l \in \{1, 2, \dots, k\}$, and define*

$$u_l = \frac{P_l u}{\sqrt{1-|\gamma|^2}}, \quad v_l = \frac{P_l T^{*l} v}{\sqrt{1-|\gamma|^2}}, \quad T_l = P_l T P_l, \quad \gamma_l = \frac{\sigma_l}{1-|\gamma|^2}.$$

*Then the function $s_l$,*

$$s_l(z) := z^{-l}\frac{s(z) - \sigma_0}{1-\sigma_0^* s(z)},$$

*belongs to the same class* $S_\kappa^0$ *and it is the characteristic function of the minimal coisometric colligation*

$$V_l = \begin{pmatrix} T_l & u_l \\ \langle \cdot, v_l \rangle & \gamma_l \end{pmatrix}$$

*in the space* $\mathscr{H}_l \oplus \mathbb{C}$.

*Proof.* We consider the colligation $\hat{V}$ from Lemma 3.2, and observe that for $j \in \{0, 1, \ldots, k-1\}$ we have $\hat{T}^j u = T^j u$ and $\hat{\sigma}_j = \sigma_j/(1-|\gamma|^2) = 0$. For the characteristic function $s_{\hat{V}}$ we find

$$\begin{aligned} s_{\hat{V}}(z) &= z^l(\hat{\sigma}_l + \hat{\sigma}_{l+1} z + \cdots) = z\langle (I - z\hat{T})^{-1}\hat{u}, \hat{v}\rangle \\ &= z^l(\langle \hat{T}^{l-1}\hat{u}, \hat{v}\rangle + z\langle \hat{T}^l \hat{u}, \hat{v}\rangle + z^2\langle \hat{T}^{l+1}\hat{u}, \hat{v}\rangle + \cdots) \\ &= z^l(\gamma_l + z\langle \hat{u}, \hat{T}^{*l}\hat{v}\rangle + z^2\langle \hat{T}\hat{u}, \hat{T}^{*l}\hat{v}\rangle + \cdots) \\ &= z^l(\gamma_l + z\langle (I - z\hat{T})^{-1}\hat{u}, \hat{T}^{*l}\hat{v}\rangle) \\ &= z^l(\gamma_l + z\langle (I - z\hat{T}_l)^{-1}\hat{u}_l, \hat{v}_l\rangle) \end{aligned}$$

with

$$\hat{u}_l := \hat{u}, \quad \hat{v}_l := \hat{T}^{*l}\hat{v}, \quad \hat{T}_l := \hat{T}, \quad \gamma_l := \hat{\sigma}_l = \frac{\sigma_l}{1-|\gamma|^2}.$$

The Lemma 3.1 implies $\hat{T}^l \mathscr{H}_l^\perp = \mathscr{H}_{l-1}^\perp \subset \mathscr{H}_l^\perp$ (with $\mathscr{H}_0^\perp := \{0\}$) and $\ker \hat{T}^l = \mathscr{H}_l^\perp$. It follows that $(I - z\hat{T})^{-1}\mathscr{H}_l^\perp = \mathscr{H}_l^\perp$, $\hat{T}^* \mathscr{H}_l \subset \mathscr{H}_l$ and $\overline{\operatorname{ran}}\, \hat{T}^{*l} = \mathscr{H}_l$ and, in particular, $P_l \hat{v}_l = P_l \hat{T}^{*l}\hat{v} = \hat{v}_l$. So we finally get

$$\begin{aligned} \langle (I - z\hat{T}_l)^{-1}\hat{u}_l, \hat{v}_l\rangle &= \langle (I - z\hat{T}_l)^{-1}\hat{u}_l, P_l\hat{v}_l\rangle \\ &= \langle (I - z\hat{T})^{-1}(P_l + (I - P_l))\hat{u}_l, P_l\hat{v}_l\rangle \\ &= \langle (I - z\hat{T})^{-1}P_l\hat{u}_l, P_l\hat{v}_l\rangle \\ &= \langle (I - zP_l\hat{T}P_l)^{-1}P_l\hat{u}_l, P_l\hat{v}_l\rangle \\ &= \langle (I - zT_l)^{-1}u_l, v_l\rangle. \end{aligned}$$

For the last equality we used that

$$P_l\hat{T}P_l = P_l\left(T + \frac{\gamma^*}{1-|\gamma|^2}\langle \cdot, v\rangle u\right)P_l = P_lTP_l = T_l,$$

$$P_l\hat{u}_l = P_l\hat{u} = \frac{1}{\sqrt{1-|\gamma|^2}}P_lu = u_l,$$

and

$$\begin{aligned}
P_l\hat{v}_l = P_l\hat{T}^{*l}\hat{v} &= \frac{1}{\sqrt{1-|\gamma|^2}}P_l\hat{T}^{*l}v \\
&= \frac{1}{\sqrt{1-|\gamma|^2}}P_l\left(T^* + \frac{\gamma}{1-|\gamma|^2}\langle\cdot,u\rangle v\right)^l v \\
&= \frac{1}{\sqrt{1-|\gamma|^2}}P_lT^{*l}v = v_l.
\end{aligned}$$

The coisometry of $V$ is proved in the following three steps in which we show the analogs of the three equations in (2.2). In each step we use that, on account of Lemma 3.2, $\hat{V}$ is coisometric.

(1) From the inclusion $\hat{T}^*\mathscr{H}_l \subset \mathscr{H}_l$ we see that $P_l\hat{T}^*P_l = \hat{T}^*P_l$ and hence

$$T_lT_l^* + \langle\cdot,u_l\rangle u_l = P_l\hat{T}P_l\hat{T}^*P_l + \langle\cdot,P_l\hat{u}\rangle P_l\hat{u} = P_l(\hat{T}\hat{T}^* + \langle\cdot,\hat{u}\rangle\hat{u})|_{\mathscr{H}_l} = I.$$

(2) Since ran $\hat{T}^{*l} = \mathscr{H}_l$ we have $P_l\hat{T}^{*l} = \hat{T}^{*l}$ and hence

$$\begin{aligned}
T_lv_l + \gamma_l^*u_l &= P_l\hat{T}P_l\hat{T}^{*l}\hat{v} + \gamma^*P_l\hat{u} \\
&= P_1(\hat{T}\hat{T}^{*l}\hat{v} + \gamma_l^*\hat{u}) \\
&= P_l(\hat{T}^{*(l-1)}\hat{v} - \langle\hat{T}^{*(l-1)}\hat{v},\hat{u}\rangle\hat{u} + \gamma_l^*\hat{u}) \\
&= (-\langle\hat{v},\hat{T}^{(l-1)}\hat{u}\rangle^* + \gamma_l^*)P_l\hat{u} = 0.
\end{aligned}$$

(3) We distinguish between $1 \leqslant l \leqslant k-1$ and $l = k$. In the first case $\gamma_l = 0$ and

$$\begin{aligned}
\langle v_l,v_l\rangle + |\gamma_l|^2 &= \langle P_l\hat{T}^{*l}\hat{v}, P_l\hat{T}^{*l}\hat{v}\rangle = \langle\hat{T}^{*l}\hat{v},\hat{T}^{*l}\hat{v}\rangle \\
&= \langle\hat{T}\hat{T}^{*l}\hat{v},\hat{T}^{*(l-1)}\hat{v}\rangle = \langle\hat{T}^{*(l-1)}\hat{v} - \langle\hat{v},\hat{T}^{(l-1)}\hat{u}\rangle\hat{u},\hat{T}^{*(l-1)}\rangle \\
&= \langle\hat{T}^{*(l-1)}v,\hat{T}^{*(l-1)}v\rangle = \cdots = \langle\hat{v},\hat{v}\rangle = 1.
\end{aligned}$$

In the second case similar calculations give

$$\langle v_k,v_k\rangle + |\gamma_k|^2 = \langle\hat{v},\hat{v}\rangle - |\langle\hat{v},\hat{T}^{(k-1)}\hat{u}\rangle|^2 + |\gamma_k|^2 = 1 - |\gamma_k|^2 + |\gamma_k|^2 = 1.$$

Finally, the minimality of $V_l$ can be derived from the minimality of $\hat{V}$ which is guaranteed by Lemma 3.2. We omit the details since the minimality of $V_l$ was shown in [2, Corollary 5.2]. □

**3.3.** Now we suppose that $|\sigma_0| > 1$. Then the steps in the proof of Theorem 3.3 can be carried out leading to a characteristic function $s_l$, which we now denote by $s_l^-$, and a colligation $V_l^-$. The main operator of this colligation is a contraction in the space $(\mathscr{H}_l)_-$. In an extra step, for $l = k$ we take the inverse function $(s_k^-(z))^{-1}$ having in its colligation a main operator which is a contraction in $\mathscr{H}_k$, and this is the desired Schur transformation. Observe that the last step cannot be carried out for $l<k$ since in this case the inverse function $(s_l^-(z))^{-1}$ has a pole at $z=0$ of order $k-l$.

**Theorem 3.4.** *Given the function $s\in S_\kappa^0$ with $|\sigma_0|>1$ and a corresponding minimal coisometric colligation $V$ in $\mathscr{H}\oplus\mathbb{C}$ as in* (3.6) *with $\gamma=\sigma_0$. Let $k$ be given by* (3.2). *Then the function $s_k(z)$,*

$$s_k(z)=z^k\frac{1-\sigma_0^*s(z)}{s(z)-\sigma_0}$$

*belongs to the class $S_{\kappa-k}^0$, and it is the characteristic function of the minimal coisometric colligation*

$$V_k=\begin{pmatrix} P_kTP_k-\dfrac{\langle\cdot,P_kT^{*k}v\rangle P_ku}{\sigma_k} & \dfrac{\sqrt{|\gamma|^2-1}}{\sigma_k}P_ku \\ \dfrac{\sqrt{|\gamma|^2-1}}{\sigma_k}\langle\cdot,P_kT^{*k}v\rangle & \dfrac{1-|\gamma|^2}{\sigma_k}\end{pmatrix}$$

*in the space $\mathscr{H}_k\oplus\mathbb{C}$.*

*Proof.* By the same reasoning as in the proof of Theorem 3.3 we find

$$\frac{s(z)-\sigma_0}{1-\sigma_0^*s(z)}=s_{\hat{V}_-}(z)=z^ls_{V_l^-}(z), \tag{3.7}$$

where $l\in\{1,2,\dots,k\}$ and $V_l^-$ is the coisometric colligation

$$V_l^-=\begin{pmatrix} P_lTP_l & \dfrac{P_lu}{\sqrt{|\gamma|^2-1}} \\ \dfrac{\langle\cdot,P_lT^{*l}v\rangle_-}{\sqrt{|\gamma|^2-1}} & \dfrac{\sigma_l}{1-|\gamma|^2}\end{pmatrix}$$

in the space $(\mathscr{H}_l)_-\oplus\mathbb{C}$. The relation (3.7) yields for $l=k$

$$z^k\frac{1-\sigma_0^*s(z)}{s(z)-\sigma_0}=\left(s_{V_k^-}(z)\right)^{-1},$$

and according to Lemma 2.1 the function on the right hand side is the characteristic function of the colligation $V_k$ in the space $\mathscr{H}_k\oplus\mathbb{C}$. For the proof of the minimality and the coisometry of the colligation $V_k$ we refer to [2, Theorem 6.1]. □

## 4. The Case $|\sigma_0|=1$

**4.1.** In this section we suppose that in the expansion (2.4) we have $|\sigma_0|=1$. As before, the integer $k(\geqslant 1)$ is determined by the relation (3.2), that is, $\sigma_k\neq 0$ and, if $k>1$, then

$$\sigma_1=\cdots=\sigma_{k-1}=0. \tag{4.1}$$

With the functions

$$\hat{s}(z):=\frac{\sigma_0z^k}{s(z)-\sigma_0},\quad \hat{s}_1(z):=\frac{\hat{s}(z)-Q(z)}{z^k},\quad \tilde{s}_1(z):=\frac{1}{z^q}\frac{\hat{s}_1(z)}{1+\hat{s}_1(z)}, \tag{4.2}$$

the Schur transformation (1.3) can be written in the form

$$s_{2k+q}(z) = \sigma_0 z^q \left(1 + \frac{z^k}{\frac{\sigma_0 z^k}{s(z)-\sigma_0} - Q(z)}\right) = \sigma_0 z^q \left(1 + \frac{1}{\frac{\hat{s}(z)-Q(z)}{z^k}}\right)$$
$$= \sigma_0 z^q \left(1 + \frac{1}{\hat{s}_1(z)}\right) = \sigma_0 \tilde{s}_1(z)^{-1}.$$

We shall find, step by step according to this formula, colligations with the characteristic functions $\hat{s}, \hat{s}_1, \tilde{s}_1^{-1}$ (only if $q > 0$) and, finally, $s_{2k+q}$.

Define

$$\hat{V} := \begin{pmatrix} \hat{T} & \hat{u} \\ \langle \cdot, \hat{v} \rangle & \hat{\gamma} \end{pmatrix} \tag{4.3}$$

with the entries

$$\begin{aligned} \hat{T} &:= T - \frac{\langle \cdot, T^{*k} v \rangle u}{\langle T^{k-1} u, v \rangle}, \quad & \hat{u} &:= \frac{\gamma\, u}{\langle T^{k-1} u, v \rangle}, \\ \hat{v} &:= -\frac{T^{*k} v}{\langle T^{k-1} u, v \rangle^*}, \quad & \hat{\gamma} &:= \frac{\gamma}{\langle T^{k-1} u, v \rangle}. \end{aligned} \tag{4.4}$$

Note that the definition of these entries here differs from the definition in Section 3 according to the different values of $\gamma$.

**Lemma 4.1.** *Let $s$ be the characteristic function of the coisometric colligation* (2.1) *and let $k$ be defined by* (3.2). *Then the function*

$$\hat{s}(z) = \frac{\sigma_0 z^k}{s(z) - \sigma_0}$$

*is the characteristic function of the colligation* (4.3).

*Proof.* We have

$$\begin{aligned} \hat{s}(z) &= \frac{\sigma_0 z^k}{s(z) - \sigma_0} \\ &= \frac{\sigma_0 z^k}{z \langle (I - zT)^{-1} u, v \rangle} \\ &= \frac{\sigma_0 z^k}{\sum_{j=k-1}^{\infty} z^{j+1} \langle T^j u, v \rangle} \\ &= \frac{\sigma_0}{\sum_{j=0}^{\infty} z^j \langle T^{k-1} T^j u, v \rangle} \\ &= \frac{\sigma_0}{\langle T^{k-1} u, v \rangle + \sum_{j=1}^{\infty} z^j \langle T^{j-1} u, T^{*k} v \rangle} \\ &= \frac{\sigma_0}{\langle T^{k-1} u, v \rangle + z \langle (I - zT)^{-1} u, T^{*k} v \rangle}. \end{aligned}$$

It remains to apply Lemma 2.1. □

In the sequel we assume that $V$ in (2.1) is coisometric and use the notation of Section 2. We often need the following relations for $m\in\{0,1,\dots,k-1\}$ and $n\in\{0,1,\dots\}$:

$$\langle T^{*m}v,T^{*n}v\rangle=\langle T^{*(m-1)}v,T^{*(n-1)}v\rangle-\langle T^{*(m-1)}v,u\rangle\langle u,T^{*(n-1)}v\rangle$$
$$=\begin{cases}\langle v,T^{*(n-m)}v\rangle=\langle Tv,T^{*(n-m-1)}v\rangle=-\sigma_0^*\sigma_{n-m}, & m<n,\\ \langle v,v\rangle=0, & m=n,\\ -\sigma_0\sigma_{m-n}^*=0, & n<m<k,\end{cases}\tag{4.5}$$

which is a special case of (2.2).

**Lemma 4.2.** *Let $s$ be the characteristic function of the coisometric colligation $V$ in* (2.1). *If $k$ is defined by* (3.2) *and $\hat{T}$ by* (4.3), (4.4), *then the following statements hold*:

1. *The subspace* $\operatorname{span}\{v,T^*v,\dots,T^{*(k-1)}v\}$ *is neutral.*
2. *The subspace* $\operatorname{span}\{v,T^*v,\dots,T^{*(2k-1)}v\}$ *is nondegenerate.*
3.

$$\hat{T}T^{*l}v=\begin{cases}0 & \text{if } l=0,\\ T^{*(l-1)}v & \text{if } l=1,\dots,2k-1,\\ T^{*(l-1)}v+\sigma_0\frac{\sigma_{l-k}^*}{\sigma_k}u & \text{if } l=2k,2k+1,\dots.\end{cases}$$

*In particular, the elements $v$, $T^*v,\dots,T^{*(2k-1)}v$ form a Jordan chain* (*of length* $2k$) *of the operator $\hat{T}$ at zero*:

$$\hat{T}v=0,\quad \hat{T}T^*v=v,\dots,\hat{T}T^{*(2k-1)}v=T^{*(2k-2)}v.$$

4. *The elements $T^{*(k-1)}v$, $T^{*(k-2)}v,\dots,v$ form a Jordan chain of length $k$ of the operator $\hat{T}^*$ at zero, that is,*

$$\hat{T}^*T^{*(k-1)}v=0,\quad \hat{T}^*T^{*(k-2)}v=T^{*(k-1)}v,\dots,\hat{T}^*v=T^*v.$$

*Proof.* The first two claims follow from (4.5) and (4.1). In order to prove 3. we first observe that

$$\hat{T}v=Tv-\frac{\langle v,T^{*k}v\rangle u}{\sigma_k}=Tv-\frac{\langle -\sigma_0^*u,T^{*(k-1)}v\rangle u}{\sigma_k}=Tv+\sigma_0^*u=0.$$

Further,

$$\begin{aligned}\hat{T}T^{*l}v&=T\,T^{*l}v-\frac{\langle T^{*l}v,T^{*k}v\rangle}{\sigma_k}u\\ &=T^{*(l-1)}v-\langle T^{*(l-1)}v,u\rangle u\\ &\quad-\frac{\langle T^{*(l-1)}v,T^{*(k-1)}v\rangle-\langle T^{*(l-1)}v,u\rangle\langle u,T^{*(k-1)}v\rangle}{\sigma_k}u\\ &=T^{*(l-1)}v-\frac{\langle T^{*(l-1)}v,T^{*(k-1)}v\rangle}{\sigma_k}u,\end{aligned}$$

and it remains to use (4.5). Finally, 4. follows from

$$\hat{T}^* T^{*(k-1)} v = T^{*k} - \frac{\langle T^{*(k-1)} v, u\rangle}{\langle v, T^{k-1} u\rangle} T^{*k} v = 0,$$

and, for $0 \leqslant j \leqslant k-2$,

$$\hat{T}^* T^{*j} v = T^{*(j+1)} v - \frac{\langle T^{*j} v, u\rangle}{\langle v, T^{k-1} u\rangle} T^{*k} v = T^{*(j+1)} v - \frac{\sigma_{j+1}^*}{\langle v, T^{k-1} u\rangle} T^{*k} v = T^{*(j+1)} v.$$

□

Next we find a colligation $\hat{V}_1$ having $\hat{s}_1(z)$ as its characteristic function. Since

$$\hat{s}(z) = \hat{\gamma} + z\langle \hat{u}, \hat{v}\rangle + z^2 \langle \hat{T}\hat{u}, \hat{v}\rangle + \cdots$$

and by Lemma 4.1, the function $Q(z)$ in the Schur algorithm (1.3) becomes

$$\begin{aligned} Q(z) = {} & \hat{\gamma} + z\langle \hat{u}, \hat{v}\rangle + z^2 \langle \hat{T}\hat{u}, \hat{v}\rangle + \cdots + z^{k-1} \langle \hat{T}^{k-2} \hat{u}, \hat{v}\rangle \\ & - z^{k+1} \langle \hat{T}^{k-2} \hat{u}, \hat{v}\rangle^* - z^{k+2} \langle \hat{T}^{k-3} \hat{u}, \hat{v}\rangle^* - \cdots - z^{2k} \hat{\gamma}^*. \end{aligned}$$

It follows that

$$\begin{aligned} \hat{s}(z) - Q(z) = z^k \{ & \langle \hat{T}^{k-1} \hat{u}, \hat{v}\rangle + z(\langle \hat{u}, \hat{T}^{*k} \hat{v}\rangle + \langle \hat{T}^{k-2} \hat{u}, \hat{v}\rangle^*) \\ & + z^2 (\langle \hat{T}\hat{u}, \hat{T}^{*k} \hat{v}\rangle + \langle \hat{T}^{k-3} \hat{u}, \hat{v}\rangle^*) + \cdots + \\ & + z^k (\langle \hat{T}^{k-1} \hat{u}, \hat{T}^{*k} \hat{v}\rangle + \hat{\gamma}^*) + z^{k+1} \langle \hat{T}^k \hat{u}, \hat{T}^{*k} \hat{v}\rangle + \cdots \}. \end{aligned} \tag{4.6}$$

We introduce the new element

$$\hat{v}_1 := \hat{T}^{*k} \hat{v} - \sum_{j=0}^{k-1} \alpha_j T^{*j} v, \tag{4.7}$$

and show that the coefficients $\alpha_j$ can be chosen such that

$$\begin{aligned} \hat{s}(z) - Q(z) & = z^k \left\{ \langle \hat{T}^{k-1} \hat{u}, \hat{v}\rangle + \sum_{j=0}^{\infty} z^{j+1} \langle \hat{T}^j \hat{u}, \hat{v}_1\rangle \right\} \\ & = z^k \{ \langle \hat{T}^{k-1} \hat{u}, \hat{v}\rangle + z\langle (I - z\hat{T})^{-1} \hat{u}, \hat{v}_1\rangle \}. \end{aligned} \tag{4.8}$$

To this end we first observe that Lemma 4.2 implies for $j = 0, 1, \ldots, k-1$ the relations

$$\hat{T}^{*k} T^{*j} v = 0, \quad \hat{T}^{*(k-l)} T^{*j} v = \begin{cases} 0 & \text{if } j \geqslant l, \\ T^{*(k-l+j)} v & \text{if } j < l. \end{cases} \tag{4.9}$$

Now it follows that

$$\langle \hat{T}^{k+l} \hat{u}, \hat{T}^{*k} \hat{v}\rangle = \langle \hat{T}^{k+l} \hat{u}, \hat{v}_1\rangle, \quad l = 0, 1, \ldots, \tag{4.10}$$

hence the terms containing $z^{2k+1}$, $z^{2k+2}, \ldots$ in (4.6) and (4.8) coincide. It remains to choose the $\alpha_j$'s such that

$$\langle \hat{T}^{k-1} \hat{u}, \hat{T}^{*k} \hat{v}\rangle + \hat{\gamma}^* = \langle \hat{T}^{k-1} \hat{u}, \hat{v}_1\rangle \tag{4.11}$$

and

$$\langle \hat{T}^{k-l}\hat{u}, \hat{T}^{*k}\hat{v}\rangle + \langle \hat{v}, \hat{T}^{l-2}\hat{u}\rangle = \langle \hat{T}^{k-l}\hat{u}, \hat{v}_1\rangle, \quad l = 2, \dots, k, \tag{4.12}$$

that is, such that

$$\hat{\gamma}^* = -\left\langle \hat{T}^{k-1}\hat{u}, \sum_{j=0}^{k-1} \alpha_j T^{*j} v \right\rangle, \quad \langle \hat{T}^{l-2}\hat{u}, \hat{v}\rangle^* = -\left\langle \hat{T}^{k-l}\hat{u}, \sum_{j=0}^{k-1} \alpha_j T^{*j} v \right\rangle.$$

By (4.9), these relations imply

$$\alpha_0 = -\frac{\sigma_0^2}{\sigma_k},$$

and

$$\begin{aligned}
\langle \hat{T}^{l-2}\hat{u}, \hat{v}\rangle^* &= -\left\langle \hat{T}^{k-l}\hat{u}, \sum_{j=0}^{k-1} \alpha_j T^{*j} v \right\rangle \\
&= -\sum_{j=0}^{l-1} \alpha_j^* \langle \hat{T}^{k-l}\hat{u}, T^{*j} v\rangle \\
&= -\sum_{j=0}^{l-2} \alpha_j^* \langle \hat{u}, T^{*(k-l+j)} v\rangle - \alpha_{l-1}^* \langle \hat{u}, T^{*(k-1)} v\rangle, \\
&= -\alpha_{l-1}^* \langle \hat{u}, T^{*(k-1)} v\rangle,
\end{aligned}$$

therefore

$$\alpha_j = -\frac{\sigma_0^2}{\sigma_k} \langle \hat{T}^{j-1} u, \hat{v}\rangle, \quad j = 1, 2, \dots, k-1.$$

Thus, finally,

$$\hat{s}_1(z) = \frac{\hat{s}(z) - Q(z)}{z^k} = \langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle + z\langle (I - z\hat{T})^{-1}\hat{u}, \hat{v}_1\rangle,$$

and the colligation $\hat{V}_1$ becomes

$$\hat{V}_1 = \begin{pmatrix} \hat{T} & \hat{u} \\ \langle \cdot, \hat{v}_1\rangle & \langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle \end{pmatrix}.$$

Now suppose first that $\hat{s}_1(0) = \langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle \neq 0$. Then the function $s_{2k}$ is given by

$$s_{2k}(z) = \sigma_0 + \frac{\sigma_0}{\hat{s}_1(z)}.$$

An application of Lemma 2.1 yields

$$s_{2k}(z) = s_{V'_{2k}}(z)$$

with

$$V'_{2k} = \begin{pmatrix} \hat{T} - \dfrac{\langle \cdot, \hat{v}_1\rangle \hat{u}}{\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle} & \dfrac{\hat{u}}{\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle} \\ -\dfrac{\langle \cdot, \hat{v}_1\rangle}{\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle}\sigma_0 & \sigma_0 + \dfrac{\sigma_0}{\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle} \end{pmatrix} \tag{4.13}$$

Thus, we have found a colligation having $s_{2k}$ as its characteristic function. This colligation, however, need not be minimal. It will be reduced to a minimal one in Subsection 4.3. Here we first find an analogous nonminimal colligation in the case that $\hat{s}_1(0) = \langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle = 0$.

We introduce for $q = 1, 2, \ldots$ the following condition:

$$(C_q) \quad \langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle = 0, \quad \langle \hat{u}, \hat{v}_1\rangle = \cdots = \langle \hat{T}^{q-2}\hat{u}, \hat{v}_1\rangle = 0, \quad \langle \hat{T}^{q-1}\hat{u}, \hat{v}_1\rangle \neq 0,$$

where $(C_1)$ means that $\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle = 0$, $\langle \hat{u}, \hat{v}_1\rangle \neq 0$.

If $(C_q)$ is satisfied we consider the function

$$\tilde{s}_1(z) := \frac{1}{z^q}\frac{\hat{s}_1(z)}{1+\hat{s}_1(z)},$$

which is up to the factor $\sigma_0$ the inverse of the function $s_{2k+q}$. According to Lemma 2.3, it is the characteristic function of the colligation $\tilde{V}_1$:

$$\tilde{V}_1 = \begin{pmatrix} \hat{T}_1 & \hat{u} \\ \langle \cdot, \hat{T}_1^{*q}\hat{v}_1\rangle & \langle \tilde{T}_1^{q-1}\hat{u}, \hat{v}_1\rangle \end{pmatrix},$$

where

$$\tilde{T}_1 := \hat{T} - \langle \cdot, \hat{v}_1\rangle\hat{u} = T - \frac{\langle \cdot, T^{*k}v\rangle u}{\sigma_k} - \langle \cdot, \hat{v}_1\rangle\hat{u}. \tag{4.14}$$

An application of Lemma 2.1 yields that $s_{2k+q}(z) = \sigma_0\tilde{s}_1(z)^{-1}$ is the characteristic function of the colligation $V'_{2k+q}$:

$$V'_{2k+q} = \begin{pmatrix} \tilde{T}_1 - \dfrac{\langle \cdot, \tilde{T}_1^{*q}\hat{v}_1\rangle\hat{u}}{\langle \tilde{T}_1^{q-1}\hat{u}, \hat{v}_1\rangle} & \dfrac{\hat{u}}{\langle \tilde{T}_1^{q-1}\hat{u}, \hat{v}_1\rangle} \\ -\dfrac{\langle \cdot, \tilde{T}_1^{*q}\hat{v}_1\rangle\sigma_0}{\langle \tilde{T}_1^{q-1}\hat{u}, \hat{v}_1\rangle} & \dfrac{\sigma_0}{\langle \tilde{T}_1^{q-1}\hat{u}, \hat{v}_1\rangle} \end{pmatrix}. \tag{4.15}$$

Hence, if $\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle = 0$, then the characteristic function of the colligation $V'_{2k+q}$ in (4.15) is the Schur transformation $s_{2k+q}$ from (1.3) of the given function $s$. However, also this colligation need not be minimal. In order to get a minimal colligation, we have to reduce the space $\mathscr{H}$.

**4.2.** First we prove some more relations between $T$ and $\hat{T}$.

**Lemma 4.3.** *The following relation holds*:

$$\hat{T}^{r-1}u = T^{r-1}u - \sum_{j=1}^{r-1}\frac{\sigma_{k+r-j}}{\sigma_k}\hat{T}^{j-1}u, \quad r = 1, 2, \ldots. \tag{4.16}$$

*Proof.* We prove (4.16) by induction. For $r = 1$ it is trivially true. If it is true for $r = l$ we get

$$
\begin{aligned}
\hat{T}^l u &= T\hat{T}^{l-1}u - \frac{\langle \hat{T}^{l-1}u, T^{*k}v\rangle u}{\sigma_k} \\
&= T\left(T^{l-1}u - \sum_{j=1}^{l-1} \frac{\sigma_{k+l-j}}{\sigma_k}\hat{T}^{j-1}u\right) \\
&\quad - \frac{1}{\sigma_k}\left\langle T^{l-1}u - \sum_{j=1}^{l-1} \frac{\sigma_{k+l-j}}{\sigma_k}\hat{T}^{j-1}u, T^{*k}v\right\rangle u \\
&= T^l u - \sum_{j=1}^{l-1} \frac{\sigma_{k+l-j}}{\sigma_k}\left(\hat{T}\hat{T}^{j-1}u + \frac{\langle \hat{T}^{j-1}u, T^{*k}v\rangle}{\sigma_k}u\right) \\
&\quad - \frac{1}{\sigma_k}\left(\langle T^{l-1}u, T^{*k}v\rangle u - \sum_{j=1}^{l-1} \frac{\sigma_{k+l-j}}{\sigma_k}\langle \hat{T}^{j-1}u, T^{*k}v\rangle u\right) \\
&= T^l u - \sum_{j=1}^{l-1} \frac{\sigma_{k+l-j}}{\sigma_k}\hat{T}^j u - \frac{\sigma_{k+l}}{\sigma_k}u \\
&= T^l u - \sum_{j=1}^{l} \frac{\sigma_{k+l-j+1}}{\sigma_k}\hat{T}^{j-1}u.
\end{aligned}
$$

□

The relation (4.16) can also be written in the form

$$
T^{r-1}u - \sum_{j=1}^{r} \frac{\sigma_{k+r-j}}{\sigma_k}\hat{T}^{j-1}u = 0, \quad r = 1, 2, \ldots, \tag{4.17}
$$

which implies that

$$
\sigma_{k+r} + \sum_{j=1}^{r} \langle \hat{T}^{j-1}u, \hat{v}\rangle \sigma_{k+r-j} = 0. \tag{4.18}
$$

**Lemma 4.4.** *The element $\hat{v}_1$ from* (4.7) *has the following properties*:

$$
\langle \hat{v}_1, T^{*l}v\rangle = 0, \quad l = 0, 1, \ldots, 2k-1. \tag{4.19}
$$

*For $s = 0, 1, \ldots,$ it holds*

$$
\begin{aligned}
\langle \hat{v}_1, T^{*(2k+s)}v\rangle &= \sigma_{k+s} + \frac{\sigma_0^*}{\sigma_k^*}\sigma_s{}' + \sum_{j=0}^{\min(k-1,s)} \sigma_{k+s-j}\langle \hat{T}^{*(k-j-1)}\hat{v}, \hat{u}\rangle \\
&\quad + \sum_{j=0}^{s} \sigma_{k+s-j}\langle \hat{T}^{k+j-1}\hat{u}, \hat{v}\rangle,
\end{aligned} \tag{4.20}
$$

*where $\sigma_0{}' = 0$ and $\sigma_s{}' = \sigma_s$ if $s > 0$.*

*Proof.* If $l = 0, 1, \ldots, k$, the relation (4.19) follows from (4.5) and Lemma 4.2. If $l = k + r$, $r = 1, \ldots, k-1$, then by (4.5), Lemma 4.2 and (4.18) we get

$$\begin{aligned}\langle \hat{v}_1, T^{*(k+r)} v\rangle &= \langle \hat{T}^{*k}\hat{v}, T^{*(k+r)} v\rangle - \sum_{j=0}^{k-1} \alpha_j \langle T^{*j} v, T^{*(k+r)} v\rangle \\ &= -\frac{\langle T^{*k} v, T^{*r} v\rangle}{\sigma_k^*} - \frac{\sigma_0}{\sigma_k}\sigma_{k+r} \\ &\quad - \sum_{j=1}^{r}\left(-\frac{\sigma_0^2}{\sigma_k}\langle \hat{T}^{j-1} u, \hat{v}\rangle\right)(-\sigma_0^*\sigma_{k+r-j}) = 0.\end{aligned}$$

It remains to prove (4.20). Using

$$\langle T^{*j} v, T^{*(2k+s)} v\rangle = -\sigma_0^*\sigma_{2k+s-j} \quad \text{if} \quad j < k,$$

and the relation (4.16), we obtain

$$\begin{aligned}&\langle \hat{v}_1, T^{*(2k+s)} v\rangle \\ &\quad = \langle \hat{T}^{*k}\hat{v}, T^{*(2k+s)} v\rangle + \frac{\sigma_0^2}{\sigma_k}\langle v, T^{*(2k+s)} v\rangle + \frac{\sigma_0^2}{\sigma_k}\sum_{j=1}^{k-1}\langle \hat{T}^{j-1} u, \hat{v}\rangle\langle T^{*j} v, T^{*(2k+s)} v\rangle \\ &\quad = \langle \hat{T}^{*k}\hat{v}, T^{*(2k+s)} v\rangle + \frac{\sigma_0^2}{\sigma_k}\langle v, T^{*(2k+s)} v\rangle - \frac{\sigma_0}{\sigma_k}\sum_{j=1}^{k+s-1}\sigma_{2k+s-j}\langle \hat{T}^{j-1} u, \hat{v}\rangle \\ &\qquad + \frac{\sigma_0}{\sigma_k}\sum_{j=k}^{k+s-1}\sigma_{2k+s-j}\langle \hat{T}^{j-1} u, \hat{v}\rangle \\ &\quad = \langle \hat{T}^{*k}\hat{v}, T^{*(2k+s)} v\rangle + \frac{\sigma_0^2}{\sigma_k}\langle v, T^{*(2k+s)} v\rangle - \sigma_0\langle T^{k+s-1} u - \hat{T}^{k+s-1} u, \hat{v}\rangle \\ &\qquad + \frac{\sigma_0}{\sigma_k}\sum_{j=k}^{k+s-1}\sigma_{2k+s-j}\langle \hat{T}^{j-1} u, \hat{v}\rangle \\ &\quad = \langle \hat{T}^{*k}\hat{v}, T^{*(2k+s)} v\rangle + \frac{\sigma_0}{\sigma_k}\sum_{j=k}^{k+s}\sigma_{2k+s-j}\langle \hat{T}^{j-1} u, \hat{v}\rangle \\ &\quad = \langle \hat{T}^{*k}\hat{v}, T^{*(2k+s)} v\rangle + \frac{\sigma_0}{\sigma_k}\sum_{j=0}^{s}\sigma_{k+s-j}\langle \hat{T}^{j+k-1} u, \hat{v}\rangle.\end{aligned}$$

The first summand in the last expression becomes with Lemma 4.2, 3.:

$$\begin{aligned}&\langle \hat{T}^{*k}\hat{v}, T^{*(2k+s)} v\rangle \\ &\quad = \langle \hat{T}^{*(k-1)}\hat{v}, T^{*(2k+s-1)} v\rangle + \frac{\sigma_0^*}{\sigma_k^*}\sigma_{k+s}\langle \hat{T}^{*(k-1)}\hat{v}, u\rangle \\ &\quad = \langle \hat{v}, T^{*(k+s)} v\rangle + \frac{\sigma_0^*}{\sigma_k^*}\sum_{j=0}^{k-1}\sigma_{k+s-j}\langle \hat{T}^{*(k-j-1)}\hat{v}, u\rangle\end{aligned}$$

$$= -\left\langle \frac{T^{*k}v}{\sigma_k^*}, T^{*(k+s)}v \right\rangle + \frac{\sigma_0^*}{\sigma_k^*} \sum_{j=0}^{k-1} \sigma_{k+s-j} \langle \hat{T}^{*(k-j-1)} \hat{v}, u \rangle$$

$$= -\frac{1}{\sigma_k^*} \left( \langle T^{*(k-1)}v, T^{*(k+s-1)}v \rangle - \sigma_k^* \sigma_{k+s} \right) + \frac{\sigma_0^*}{\sigma_k^*} \sum_{j=0}^{k-1} \sigma_{k+s-j} \langle \hat{T}^{*(k-j-1)} \hat{v}, u \rangle$$

$$= \sigma_{k+s} + \frac{\sigma_0^*}{\sigma_k^*} \sigma_s{}' + \frac{\sigma_0^*}{\sigma_k^*} \sum_{j=0}^{k-1} \sigma_{k+s-j} \langle \hat{T}^{*(k-j-1)} \hat{v}, u \rangle,$$

and (4.20) follows. □

**Corollary 4.5.** *Under the assumption* $(C_q)$ *and with* $\hat{v}_1$ *in* (4.7) *we have*

$$\langle \hat{v}_1, T^{*(2k+s)}v \rangle = \sigma_{k+s}, \quad s = 0, 1, \ldots, q-1.$$

*Proof.* Formula (4.20) can be rewritten as

$$\langle \hat{v}_1, T^{*(2k+s)}v \rangle = \sigma_{k+s} + \frac{\sigma_0^*}{\sigma_k^*} \sigma_s{}' + \sum_{j=0}^{k-1} \sigma_{k+s-j} \left( \langle \hat{v}, \hat{T}^{(k-j-1)} \hat{u} \rangle + \langle \hat{T}^{(k+j-1)} \hat{u}, \hat{v} \rangle \right)$$
$$+ \sum_{j=k}^{s} \sigma_{k+s-j} \langle \hat{T}^{k+j-1} \hat{u}, \hat{v} \rangle.$$

In the first sum the summand corresponding to $j = 0$ is zero since $\langle \hat{T}^{(k-1)} \hat{u}, \hat{v} \rangle = 0$. The other summands, because of (4.12), are equal to $\sigma_{k+s-j} \langle \hat{T}^{j-1} \hat{u}, \hat{v}_1 \rangle$, $j = 1, 2, \ldots, k-1$. If $s \geqslant k$, by (4.10) and (4.11), the second sum equals:

$$\sigma_s \langle \hat{T}^{2k-1} \hat{u}, \hat{v} \rangle + \sum_{j=k+1}^{s} \sigma_{k+s-j} \langle \hat{T}^{j-1} \hat{u}, \hat{v}_1 \rangle$$
$$= -\sigma_s \hat{\gamma}^* + \sum_{j=k}^{s} \sigma_{k+s-j} \langle \hat{T}^{j-1} \hat{u}, \hat{v}_1 \rangle.$$

Thus

$$\langle \hat{v}_1, T^{*(2k+s)}v \rangle = \sigma_{k+s} + \frac{\sigma_0^*}{\sigma_k^*} (\sigma_s{}' - \sigma_s{}'') + \sum_{j=1}^{s} \sigma_{k+s-j} \langle \hat{T}^{(j-1)} \hat{u}, \hat{v}_1 \rangle,$$

where $\sigma_s'' = 0$ if $s < k$ and $= \sigma_s$ if $s \geqslant k$, that is, $\sigma'' = \sigma'$. The corollary now easily follows. □

**Lemma 4.6.** *Let* $\hat{v}_1$ *and* $\tilde{T}_1$ *be as in* (4.7) *and* (4.14). *Under the assumption* $(C_q)$, *with* $\hat{v}_1$ *from* (4.7) *it holds*

$$\langle T^{*s}v, \tilde{T}_1^{*q} \hat{v}_1 \rangle = 0, \quad s = 0, 1, \ldots, 2k+q-1. \tag{4.21}$$

*Proof.* To prove (4.21) for $s = 0, \ldots, 2k-1$, we make an induction with respect to $q$ and to $k$. For $q = 0$ the relations (4.21) coincide with (4.19). For

$1 \leqslant j \leqslant q$ we have by the induction assumption

$$\begin{aligned}\langle v, \tilde{T}_1^{*j}\hat{v}_1\rangle &= \langle \tilde{T}_1 v, \tilde{T}_1^{*(j-1)}\hat{v}_1\rangle \\ &= \langle \hat{T}v - \langle v, \hat{v}_1\rangle \hat{u}, \tilde{T}_1^{*(j-1)}\hat{v}_1\rangle \\ &= -\langle v, \hat{v}_1\rangle\langle \hat{u}, \tilde{T}_1^{*(j-1)}\hat{v}_1\rangle = 0,\end{aligned}$$

and for $1 \leqslant i \leqslant 2k-1$,

$$\begin{aligned}\langle T^{*i}v, \tilde{T}_1^{*j}\hat{v}_1\rangle &= \langle \tilde{T}_1 T^{*i}v, \tilde{T}_1^{*(j-1)}\hat{v}_1\rangle \\ &= \langle \hat{T}T^{*i}v - \langle T^{*i}v, \hat{v}_1\rangle \hat{u}, \tilde{T}_1^{*(j-1)}\hat{v}_1\rangle \\ &= \langle T^{*(i-1)}v, \tilde{T}_1^{*(j-1)}\hat{v}_1\rangle - \langle T^{*i}v, \hat{V}_1\rangle\langle \hat{u}, \tilde{T}_1^{*(j-1)}\hat{v}_1\rangle = 0,\end{aligned}$$

where we have used Lemma 4.2, 3., and the last expression vanishes because of the induction assumption. Next, for $r = 0, 1, \ldots, q-1$,

$$\begin{aligned}\langle T^{*(2k+r)}v, \tilde{T}_1^{*q}\hat{v}_1\rangle &= \langle \tilde{T}_1 T^{*(2k+r)}v, \tilde{T}_1^{*(q-1)}\hat{v}_1\rangle \\ &= \left\langle TT^{*(2k+r)}v - \frac{\langle T^{*(2k+r)}v, T^{*k}v\rangle}{\sigma_k}u - \langle T^{*(2k+r)}v, \hat{v}_1\rangle \hat{u}, \tilde{T}_1^{*(q-1)}\hat{v}_1\right\rangle \\ &= \langle T^{*(2k+r-1)}v, \tilde{T}_1^{*(q-1)}\hat{v}_1\rangle - \langle T^{*(2k+r-1)}v, u\rangle\langle u, \tilde{T}_1^{*(q-1)}\hat{v}_1\rangle \\ &\quad - \frac{\langle T^{*(2k+r-1)}v, T^{*(k-1)}v\rangle - \langle T^{*(2k+r-1)}v, u\rangle\langle u, T^{*(k-1)}v\rangle}{\sigma_k}\langle u, \tilde{T}_1^{*(q-1)}\hat{v}_1\rangle \\ &\quad - \langle T^{*(2k+r)}v, \hat{v}_1\rangle \frac{\sigma_0\langle u, \tilde{T}_1^{*(q-1)}\hat{v}_1\rangle}{\sigma_k} \\ &= \langle T^{*(2k+r-1)}v, \tilde{T}_1^{*(q-1)}\hat{v}_1\rangle \\ &\quad - \frac{1}{\sigma_k}\left(\langle T^{*(2k+r-1)}v, T^{*(k-1)}v\rangle + \sigma_0\langle T^{*(2k+r)}v, \hat{v}_1\rangle\right)\langle u, \tilde{T}_1^{*(q-1)}\hat{v}_1\rangle.\end{aligned}$$

The second summand vanishes because of

$$\langle T^{*(2k+r-1)}v, T^{*(k-1)}v\rangle = -\sigma_0\sigma_{k+r}^*,$$

see (4.5), and Corollary 4.5, the first summand vanishes by an induction argument with respect to $q$. □

**4.3.** We reduce the colligations $V'_{2k}$ and $V'_{2k+q}$ from (4.13) and (4.15) to minimal ones. To this end we introduce the spaces

$$\mathscr{L}_{2k} := \operatorname{span}\{v,\ T^*v, \ldots, T^{*(2k-1)}v\},$$
$$\mathscr{L}_{2k+q} := \operatorname{span}\{v,\ T^*v, \ldots, T^{*(2k+q-1)}v\},$$

and

$$\mathscr{H}_1 := \mathscr{L}_{2k}^{\perp}, \quad \mathscr{H}_1^q := \mathscr{L}_{2k+q}^{\perp}.$$

The relations (4.5) imply that for the space $\mathscr{L}_{2k}$ the negative and the positive index are equal to $k$, hence $\mathscr{L}_{2k}$ and $\mathscr{H}_1$ are nondegenerate, and the negative index of $\mathscr{H}_1$ is $\kappa - k$. Denote the orthogonal projection onto $\mathscr{H}_1$ by $P_1$.

Now we suppose that the condition

$$\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle \neq 0$$

is satisfied and consider $V'_{2k}$ from (4.13). Lemma 4.2, 3., (4.19) and (4.4) imply

$$P_1\hat{T}^j = (P_1\hat{T}P_1)^j = (P_1TP_1)^j, \quad j = 1, 2, \ldots,$$

$$\begin{aligned}
\hat{v}_1 = P_1\hat{v}_1 = P_1\hat{T}^{*k}\hat{v} &= -P_1\hat{T}^{*k}\frac{T^{*k}v}{\sigma_k^*} \\
&= -P_1\left(T^* - \frac{\langle\cdot, u\rangle T^{*k}v}{\sigma_k^*}\right)\hat{T}^{*(k-1)}\frac{T^{*k}v}{\sigma_k^*} \\
&= -P_1T^*\hat{T}^{*(k-1)}\frac{T^{*k}v}{\sigma_k^*} = \cdots = -\frac{P_1T^{*(2k)}v}{\sigma_k^*},
\end{aligned}$$

and we obtain

$$\begin{aligned}
\langle (I - z\hat{T})^{-1}\hat{u}, \hat{v}_1\rangle = \langle (I - z\hat{T})^{-1}\hat{u}, P_1\hat{v}_1\rangle &= \sum_{j=0}^{\infty} z^j\langle P_1\hat{T}^j\hat{u}, P_1\hat{v}_1\rangle \\
&= -\sum_{j=0}^{\infty} z^j\left\langle (P_1TP_1)^j\hat{u}, \frac{P_1T^{*(2k)}v}{\sigma_k^*}\right\rangle \\
&= -\sigma_0\left\langle (I - zP_1TP_1)^{-1}\frac{P_1u}{\sigma_k}, \frac{P_1T^{*(2k)}v}{\sigma_k^8}\right\rangle.
\end{aligned}$$

It remains to apply Lemma 2.1, and the following theorem is proved, except for the last statement.

**Theorem 4.7.** *Assume that for $s \in S_\kappa^0$ we have $|\sigma_0| = 1$, and that $q = 0$. If $V$ is a minimal coisometric colligation in the space $\mathscr{H} \oplus \mathbb{C}$ such that $s_V(z) = s(z)$ then the function*

$$s_{2k}(z) = \frac{(Q(z) - z^k)s(z) - \sigma_0 Q(z)}{\sigma_0^* Q(z)s(z) - (Q(z) + z^k)}$$

*belongs to the class $S_{\kappa-k}^0$, and it is the characteristic function of the colligation*

$$V_{2k} := \begin{pmatrix} P_1TP_1 + \dfrac{\sigma_0\langle\cdot, P_1T^{*2k}v\rangle P_1u}{\langle\hat{T}^{k-1}\hat{u}, \hat{v}\rangle\langle T^{k-1}u, v\rangle^2} & -\dfrac{\sigma_0P_1u}{\langle\hat{T}^{k-1}\hat{u}, \hat{v}\rangle\langle T^{k-1}u, v\rangle} \\ -\dfrac{\sigma_0\langle\cdot, P_1T^{*2k}v\rangle}{\langle\hat{T}^{k-1}\hat{u}, \hat{v}\rangle\langle T^{k-1}u, v\rangle} & \sigma_0 + \dfrac{\sigma_0}{\langle\hat{T}^{k-1}\hat{u}, \hat{v}\rangle} \end{pmatrix}$$

*in the space $\mathscr{H}_1 \oplus \mathbb{C}$. This colligation is minimal and coisometric.*

The theorem coincides with [2, Theorem 7.1]; indeed, the number $t_{2k}$ in that theorem can be written as $t_{2k} = -\sigma_k\langle\hat{T}^{k-1}\hat{u}, \hat{v}\rangle/\sigma_0$. The minimality and coisometry of $V_{2k}$ can be proved as in [2, Theorem 7.1].

Now we suppose that

$$\langle \hat{T}^{k-1}\hat{u}, \hat{v}\rangle = 0.$$

Then for some $q \geqslant 1$ the condition $(C_q)$ is satisfied. For the convenience of the reader we repeat it here:

$$(C_q)\ \langle \hat{T}^{k-1}\hat{u},\hat{v}\rangle = 0,\ \langle \hat{u},\hat{v}_1\rangle = \langle \hat{T}\hat{u},\hat{v}_1\rangle = \cdots = \langle \hat{T}^{q-2}\hat{u},\hat{v}_1\rangle = 0,\ \langle \hat{T}^{q-1}\hat{u},\hat{v}_1\rangle \neq 0,$$

where $\hat{v}_1$ is given by (4.7). It is easy to see that the equalities in the middle are equivalent to

$$\langle \hat{u}, \hat{v}_1\rangle = \langle T\hat{u}, \hat{v}_1\rangle = \cdots = \langle T^{q-2}\hat{u}, \hat{v}_1\rangle = 0. \tag{4.22}$$

We prove some more properties of the operator $\tilde{T}_1$ (given by (4.14)). By $P_1^q$ we denote the orthogonal projection $\mathscr{H}_1^q$

**Lemma 4.8.** *It the condition $(C_q)$ is satisfied, the following statements hold*:

1. *The elements $v, T^*v, \ldots, T^{*(2k+q-1)}v$ form a Jordan chain of the operator $\tilde{T}_1$ at zero*:

$$\tilde{T}_1 v = 0, \quad \tilde{T}_1 T^* v = v, \ldots, \tilde{T}_1 T^{*(2k+q-1)} v = T^{*(2k+q-2)} v.$$

2. *The elements $\tilde{T}_1^{*s}\hat{v}_1$, $s = 0, 1, \ldots, q-1$, are orthogonal to the space $\mathscr{L}_{2k}$ and belong to the space $\mathscr{L}_{2k+q}$.*

3.

$$P_1^q \tilde{T}_1^{*q}\hat{v}_1 = -\frac{P_1^q T^{*(2k+q)} v}{\sigma_k^*}.$$

*Proof.* 1. Firstly, by (4.19) and Lemma 4.2, 3.,

$$\tilde{T}_1 v = \hat{T} v - \langle v, \hat{v}_1\rangle \hat{u} = 0.$$

If $1 \leqslant j \leqslant 2k+q-1$ we have on account of (4.5)

$$\begin{aligned}
\hat{T}T^{*j}v &= TT^{*j}v - \frac{\langle T^{*j}v, T^{*k}v\rangle}{\sigma_k} u \\
&= T^{*(j-1)}v - \langle T^{*(j-1)}v, u\rangle u - \frac{\langle T^{*j}v, T^{*k}v\rangle}{\sigma_k} u \\
&= T^{*(j-1)}v + \frac{\sigma_0 \sigma_{j-k}^*}{\sigma_k} u
\end{aligned}$$

Now, again by (4.19) for $1 \leqslant j \leqslant 2k-1$,

$$\tilde{T}_1 T^{*j} v = \hat{T}T^{*j}v - \langle T^{*j}v, \hat{v}_1\rangle \hat{u} = T^{*(j-1)} v \in \mathscr{L}_{2k+q-1},$$

and by (4.20) with $s = 0$

$$\begin{aligned}
\tilde{T}_1 T^{*(2k)} v &= \hat{T}T^{*(2k)}v - \langle T^{*(2k)}v, \hat{v}_1\rangle \hat{u} \\
&= T^{*(2k-1)}v + \sigma_0 \frac{\sigma_k^*}{\sigma_k} u - \frac{\langle T^{*(2k)}v, \hat{v}_1\rangle \sigma_0}{\sigma_k} u \\
&= T^{*(2k-1)}v,
\end{aligned}$$

and for $1 \leqslant s \leqslant q-1$

$$\begin{aligned}\tilde{T}_1 T^{*(2k+s)} v &= \hat{T} T^{*(2k+s)} v - \langle T^{*(2k+s)} v, \hat{v}_1 \rangle \hat{u} \\ &= T^{*(2k+s-1)} v + \sigma^*_{k+s} \frac{\sigma_0}{\sigma_k} u - \sigma^*_{k+s} \frac{\sigma_0}{\sigma_k} u \\ &= T^{*(2k+s-1)} v.\end{aligned}$$

2. The first claim follows from 1. and (4.19), the proof of the second claim is left to the reader.

3. This relation is obtained as follows:

$$\begin{aligned}P_1^q \tilde{T}_1^{*q} \hat{v}_1 &= P_1^q \left( T^* - \frac{\langle \cdot, u \rangle}{\sigma_k^*} T^{*k} v - \langle \cdot, \hat{u} \rangle \hat{v}_1 \right) \tilde{T}_1^{*(q-1)} \hat{v}_1 \\ &= P_1^q T^* \left( T^* - \frac{\langle \cdot, u \rangle}{\sigma_k^*} T^{*k} v - \langle \cdot, \hat{u} \rangle \hat{v}_1 \right) \tilde{T}_1^{*(q-2)} \hat{v}_1 = \cdots \\ &= P_1^q T^{*(q-1)} \left( T^* - \frac{\langle \cdot, u \rangle}{\sigma_k^*} T^{*k} v - \langle \cdot, \hat{u} \rangle \hat{v}_1 \right) \hat{v}_1 \\ &= P_1^q T^{*q} \hat{v}_1 = P_1^q T^{*q} \left( -\hat{T}^{*k} \frac{T^{*k} v}{\sigma_k^*} - \sum_{j=0}^{k-1} \alpha_j T^{*j} v \right) \\ &= -\frac{P_1^q T^{*(2k+q)} v}{\sigma_k^*}.\end{aligned}$$

□

The following lemma implies that the space $\mathscr{L}_{2k+q}$ has positive index $k$ and negative index $k+q$.

**Lemma 4.9.** *For $r$, $s = 0, 1, \ldots, q-1$ the relations*

$$\langle \tilde{T}_1^{*r} \hat{v}_1, \tilde{T}_1^{*s} \hat{v}_1 \rangle = -\delta_{rs}$$

*hold, where $\hat{v}_1$ and $\tilde{T}_1$ are given by* (4.7) *and* (4.14).

*Proof.* For $1 \leqslant j \leqslant k-1$ the relation

$$\langle T^{*k} v, T^{*(k-j)} \hat{v} \rangle = \langle T^{*j} v, \hat{v} \rangle = -\frac{\langle T^{*j} v, T^{*k} v \rangle}{\sigma_k} = 0$$

holds (see (4.5)). We leave it to the reader to show that for $1 \leqslant l \leqslant q-1$

$$T^l \hat{v}_1 \in \operatorname{span}\{\mathscr{L}_{2k}, u, \ldots, T^{l-1} u\}. \tag{4.23}$$

The space on the right hand side is orthogonal to $\hat{v}_1$ because of (4.19) and (4.22). Now,

$$\begin{aligned}\langle \hat{v}_1, \hat{v}_1 \rangle = \langle \hat{v}_1, \hat{T}^{*k} \hat{v} \rangle &= \left\langle \hat{T}^{*k} \hat{v} - \sum_{j=0}^{k-1} \alpha_j T^{*j} v, \hat{T}^{*k} \hat{v} \right\rangle = \langle \hat{T}^{*k} \hat{v}, \hat{T}^{*k} \hat{v} \rangle \\ &= \langle T T^* \hat{T}^{*(k-1)} \hat{v}, \hat{T}^{*(k-1)} \hat{v} \rangle = \langle \hat{T}^{*(k-1)} \hat{v}, \hat{T}^{*(k-1)} \hat{v} \rangle\end{aligned}$$

$$= \left\langle \left( T^* - \frac{\langle\cdot, u\rangle T^{*k} v}{\sigma_k^*} \right) \hat{T}^{*(k-2)} \hat{v}, \hat{T}^{*(k-1)} \hat{v} \right\rangle$$
$$= \langle T^* \hat{T}^{*(k-2)} \hat{v}, \hat{T}^{*(k-1)} \hat{v} \rangle - \frac{1}{\sigma_k^*} \langle \hat{T}^{*(k-2)} \hat{v}, u \rangle \langle T^{*k} v, \hat{T}^{*(k-1)} \hat{v} \rangle$$
$$= \langle \hat{T}^{*(k-2)} \hat{v}, \hat{T}^{*(k-2)} \hat{v} \rangle = \cdots = \langle \hat{v}, \hat{v} \rangle$$
$$= \left\langle \frac{T^{*k} v}{\sigma_k^*}, \frac{T^{*k} v}{\sigma_k^*} \right\rangle = \left\langle \frac{T^{*(k-1)} v}{\sigma_k^*}, \frac{T^{*(k-1)} v}{\sigma_k^*} \right\rangle - 1 = -1.$$

For arbitrary $s = 1, 2, \ldots, q-1$ we obtain, using again $(C_q)$,

$$\langle \tilde{T}_1^{*s} \hat{v}_1, \tilde{T}_1^{*s} \hat{v}_1 \rangle = \langle T_1^{*s} \hat{v}_1, T_1^{*s} \hat{v}_1 \rangle$$
$$= \langle \hat{v}_1, \hat{v}_1 \rangle - |\langle \hat{v}_1, u \rangle|^2 = -1.$$

Finally, if $0 < r < q-1$ and $0 \leqslant s < r$ we obtain with $l = r - s$:

$$\langle \tilde{T}_1^{*r} \hat{v}_1, \tilde{T}_1^{*s} \hat{v}_1 \rangle = \langle \tilde{T}_1^{*l} \hat{v}_1, \hat{v}_1 \rangle = \langle \hat{v}_1, \tilde{T}_1^{l} \hat{v}_1 \rangle$$
$$= \left\langle \hat{v}_1, \left( T - \langle\cdot, \hat{v}_1\rangle \hat{u} - \frac{\langle\cdot, T^{*k} v\rangle u}{\sigma_k} \right) \tilde{T}_1^{l-1} \hat{v}_1 \right\rangle$$
$$= \langle \hat{v}_1, T \tilde{T}_1^{l-2} \hat{v}_1 \rangle = \cdots = \langle \hat{v}_1, T^l \hat{v}_1 \rangle,$$

and this is zero because of the remark following (4.23). □

The Lemma 4.8 1. implies that $\tilde{T}_1 \mathscr{L}_{2k+q} \subset \mathscr{L}_{2k+q}$. As above it follows now that

$$P_1^q \tilde{T}_1 = P_1^q \tilde{T}_1 P_1^q = P_1^q T P_1^q,$$
$$P_1^q \tilde{T}_1^l = (P_1^q T P_1^q)^l, \quad l = 2, 3, \ldots,$$

and we have proved the following theorem, except for the last statement.

**Theorem 4.10.** *Assume that for $s \in S_\kappa^0$ we have $|\sigma_0| = 1$, and that $q > 0$. If $V$ is a minimal coisometric colligation in the space $\mathscr{H} \oplus \mathbb{C}$ such that $s_V(z) = s(z)$ then the function*

$$s_{2k+q}(z) = \frac{1}{z^q} \frac{(Q(z) - z^k)s(z) - \sigma_0 Q(z)}{\sigma_0^* Q(z) s(z) - (Q(z) + z^k)}$$

*belongs to the class $S_{\kappa-k-q}^0$, and it is the characteristic function of the colligation*

$$V_{2k+q} := \begin{pmatrix} P_1^q T P_1^q + \dfrac{\langle\cdot, P_1^q T^{*(2k+q)} v\rangle \hat{u}}{\sigma_k \langle \tilde{T}_1^{q-1} \hat{u}, \hat{v}_1 \rangle} & \dfrac{P_1^q \hat{u}}{\langle \tilde{T}_1^{q-1} \hat{u}, \hat{v}_1 \rangle} \\ \dfrac{\sigma_0 \langle\cdot, P_1^q T^{*(2k+q)} v\rangle}{\sigma_k \langle \tilde{T}_1^{q-1} \hat{u}, \hat{v}_1 \rangle} & \dfrac{\sigma_0}{\langle \tilde{T}_1^{q-1} \hat{u}, \hat{v}_1 \rangle} \end{pmatrix}$$

*in the space $\mathscr{H}_1^q \oplus \mathbb{C}$. This colligation is minimal and coisometric.*

This theorem coincides with [2, Theorem 8.1]: the number $\tau$ there is equal to $\frac{\sigma_0}{\langle \tilde{T}_1^{q-1} \hat{u}, \hat{v}_1 \rangle}$. The minimality and coisometry of $V_{2k+q}$ then follow from [2, Theorem 8.1].

## 5. The Sequence of Schur Determinants

Consider again a function $s \in S^0$, and let

$$s(z) = \sum_{n=0}^{\infty} \sigma_n z^n \tag{5.1}$$

be its Taylor expansion at $x = 0$. We consider the matrices

$$\mathscr{A}_{j-1} = \begin{pmatrix} \sigma_0 & 0 & 0 & \dots & 0 \\ \sigma_1 & \sigma_0 & 0 & \dots & 0 \\ \sigma_2 & \sigma_1 & \sigma_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{j-1} & \sigma_{j-2} & \sigma_{j-3} & \dots & \sigma_0 \end{pmatrix},$$

and

$$\mathscr{G}_{j-1} = \mathscr{I}_j - \mathscr{A}_{j-1}\mathscr{A}_{j-1}^*,$$

where $\mathscr{I}_j$ stands for the $j \times j$ identity matrix. The matrix $\mathscr{G}_{j-1}$ is the Gram matrix for the space $\mathscr{L}_j = \operatorname{span}\{v, T^*v, \dots, T^{*(j-1)}v\}$, and hence it determines the signature of this space. Further, set

$$\delta_{j-1}(s) := \det \mathscr{G}_{j-1}, \quad j = 1, 2, \dots.$$

In [5] it is shown that the series in (5.1) defines a function $s$ in the class $S^0$ if and only if there is a natural number $n_0$ such that either $\delta_n(s) = 0$ for all $n \geqslant n_0$ or $\delta_n(s)\delta_{n+1}(s) > 0$ for all $n \geqslant n_0$. The case $\delta_{n_0-1}(s) \neq 0$ and $\delta_n(s) = 0$ for all $n \geqslant n_0$ holds if and only if there exist complex numbers $c$ and $\alpha_1, \dots, \alpha_{n_0}$ with $|c| = 1$ and $\alpha_i \alpha_j^* \neq 1$ such that

$$s(z) = c \prod_{j=1}^{n_0} \frac{z - \alpha_j}{1 - \alpha_j^* z}.$$

In the following we show that the number of negative eigenvalues of the matrix $\mathscr{G}_n$, if it is invertible, is determined through the number of sign changes of the sequence

$$1, \delta_0(s), \delta_1(s), \dots, \delta_n(s),$$

provided the zero entries in this sequence located between two nonzero entries are replaced by nonzero numbers following the simple rules for Toeplitz matrices. Although the matrix $\mathscr{G}_n$ is not Toeplitz these rules can be applied here. In order to see this, following [3] we introduce for $n = 0, 1, 2, \dots$ the matrices

$$\mathscr{M}_n = \mathscr{B}_n \operatorname{diag}(1 \ \mathscr{G}_n)\mathscr{B}_n^*, \tag{5.2}$$

where

$$\mathscr{B}_n := \begin{pmatrix} c_0 & 0 & \dots & 0 \\ c_1 & c_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n+1} & c_n & \dots & c_0 \end{pmatrix},$$

and the $c_k$ are determined recursively by the relations

$$c_0 = 1, \quad c_k = \sum_{j=0}^{k-1} c_j \sigma_{k-1-j}, \quad k = 1, 2, \ldots, n+1.$$

It was shown in [3] that $\mathscr{M}_n$ is a hermitian Toeplitz matrix. Since $c_0 = 1$ we have

$$\det \mathscr{M}_n = \delta_n(s), \quad n = 0, 1, \ldots.$$

Let

$$\varepsilon_n := \operatorname{sign} \delta_n(s) \quad \text{if} \quad \delta_n(s) \neq 0. \tag{5.3}$$

If, for example, $\delta_{h-1}(s) \neq 0$, $\delta_h(s) = \delta_{h+1}(s) = \cdots = \delta_{h+p-1}(s) = 0$, $\delta_{h+p}(s) \neq 0$, where $p > 0$, then since $\mathscr{M}_n$ is a hermitian Toeplitz matrix the number $p$ is odd (see [12, Lemma 16.1]) and we have

$$\varepsilon_{h+p} = (-1)^{\frac{p+1}{2}} \varepsilon_{h-1}$$

(see [12, Corollary 2 to Lemma 16.1]). For the indices $h, h+1, \ldots, h+p-1$ with vanishing determinants we define the numbers (see [12, Formula (16.7)])

$$\varepsilon_{h+j-1} := (-1)^{\frac{j(j-1)}{2}} \varepsilon_{h-1}, \quad j = 1, \ldots, p. \tag{5.4}$$

Now the following theorem is an immediate consequence of [12, Theorem 16.2]. For a matrix $\mathscr{B}$ by $\kappa_-(\mathscr{B})$ we denote the number of its negative eigenvalues.

**Theorem 5.1.** *If for* $0 \leqslant m < n$ *the matrices* $\mathscr{G}_m$, $\mathscr{G}_n$ *are invertible and the numbers* $\varepsilon_j$, $j = 0, 1, \ldots, n$, *are defined according to* (5.3), (5.4), *then* $\kappa_-(\mathscr{G}_n)$ *equals the number of sign changes in the sequence* $1, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$, *and the difference* $\kappa_-(\mathscr{G}_n) - \kappa_-(\mathscr{G}_m)$ *equals the number of sign changes in the sequence* $\varepsilon_m, \varepsilon_{m+1}, \ldots, \varepsilon_n$.

For the function $s \in S^0$ we denote by $J(s)$ the set of the indices $j \in \mathbb{N}$ for which the Schur transformation $s_j$ is defined. The set $J(s)$ cannot be characterized through the sequence of the $\delta_n(s)$ only. However, we prove the following theorem.

**Theorem 5.2.** *Let* $s \in S^0$ *and assume that* $h \in J(s)$, $h \geqslant 1$. *Then* $\delta_{h-1}(s) \neq 0$, *and the following holds*:

(i) *If* $\delta_h \delta_{h-1} > 0$, *then for* $s_h(z)$ *case* (1.1) *of the Schur transformation applies*;
(ii) *if* $\delta_h \delta_{h-1} < 0$, *then for* $s_h(z)$ *case* (1.2) *applies*;
(iii) *if* $\delta_h(s) = \delta_{h+1}(s) = \cdots = \delta_{h+p-1}(s) = 0$, $\delta_{h+p}(s) \neq 0$, *then for* $s_h(z)$ *case* (1.3) *applies with* $k = \frac{p+1}{2}$.

*Proof.* The first claim follows from the fact that in all cases (1.1)–(1.3) of the Schur transformation the subspace $\mathscr{L}_h$ is nondegenerate; see [2, Theorem 9.1].

Next we prove (iii), the statements (i) and (ii) follow in the same way. First we observe that the assumptions imply that the space $\mathscr{L}_h$ is nondegenerate, the spaces $\mathscr{L}_{h+1}, \mathscr{L}_{h+2}, \ldots, \mathscr{L}_{h+p}$ are degenerate and $\mathscr{L}_{h+p+1}$ is nondegenerate.

Since $\delta_{h-1} \neq 0$ and $\delta_h = 0$, we have $h = \operatorname{rank} \mathscr{G}_{h-1} = \operatorname{rank} \mathscr{G}_h$. Since $\delta_{h+p} \neq 0$ and $\delta_{h+p-1} = 0$, we have $\operatorname{rank} \mathscr{G}_{h+p} = h + p + 1$ and $\operatorname{rank} \mathscr{G}_{h+p-1} = h + p - 1$, see [12, Corollary to Lemma 6.1]. Now we use that $\operatorname{rank} \mathscr{G}_j$ is nondecreasing as a function of $j$ and conclude that there is a largest number $k'$ such that $\operatorname{rank} \mathscr{G}_{h+p-k'} > h$. In view of (5.2) we may treat the matrices $\mathscr{G}_n$ as if they were Toeplitz matrices, therefore $k'$ can be viewed as the characteristic of $\mathscr{G}_{h+p-1}$ (see [12, (14.3)]) and applying [12, Theorem 15.1 and (15.8)] we obtain $h + p - 1 = h + 2k'$. Hence $k' = (p-1)/2$, and, on account of [12, Theorem 15.6],

$$\operatorname{rank} \mathscr{G}_h = \operatorname{rank} \mathscr{G}_{h+1} = \cdots = \operatorname{rank} \mathscr{G}_{h+k-1} = h \tag{5.5}$$

and

$$\operatorname{rank} \mathscr{G}_{h+k+j-1} = h + 2j, \quad j = 1, 2, \ldots, k,$$

where we have set $k := k' + 1 = (p+1)/2$. By (5.5), $\mathscr{L}_{h+k}$ contains a $k$-dimensional isotropic subspace. The elements $T^{*h}v$, $T^{*(h+1)}v, \ldots, T^{*(h+k-1)v}$ span a complement of $\mathscr{L}_h$ in $\mathscr{L}_{h+k}$. Therefore the orthogonal projections (modulo a multiplicative constant)

$$\hat{v}, \hat{T}^{*}\hat{v}, \ldots, \hat{T}^{*(k-1)}\hat{v} \tag{5.6}$$

of the elements $T^{*h}v$, $T^{*(h+1)}v, \ldots, T^{*(h+k-1)}v$ onto the orthogonal complement of $\mathscr{L}_h$ in $\mathscr{L}_{h+p+1} = \mathscr{L}_{h+2k}$ also span a neutral subspace. On the other hand the elements in (5.6) are the first $k$ basis elements of the state space of the colligation corresponding to $s_h$, see [2, Theorem 9.1] and its proof. Hence $|s_h(0)| = 1$ and the coefficient of $z^k$ is the first nonzero coefficient with index $\geqslant 1$ in the Taylor expansion of $s_h(z)$ around $z = 0$. □

We mention that the numbers $k$ in the case of (1.2) and $q$ in the case of (1.3) can, apparently, not be determined from the sequence $(\delta_n(s))$.

As a consequence of this theorem the following relations between the set $J(s)$ of all integers $j$ such that the Schur transformation $s_j$ is defined and the sequence of the numbers $\delta_{j-1}(s)$, $j = 1, 2, \ldots$, can be formulated. If $j \in \mathbb{N}$ is such that $\delta_{j-1}(s) \neq 0$, $\delta_j(s) = \cdots = \delta_{j+p-1}(s) = 0$, $\delta_{j+p}(s) \neq 0$ then $j \in J(s)$, $j+1, j+2, \ldots, j+p \notin J(s)$ and $j+p+1$ can belong to $J(s)$ or not. In any case the function $s_{j+p+1}$ can be formed according to [2, Remark 6.2]: If it has a pole at zero then $j+p+1 \notin J(s)$, otherwise, if it is holomorphic at zero, then $j+p+1 \in J(s)$. If $\delta_{j+p+1}(s) = 0$, then the next index $j+p+2$ does not belong to $J(s)$. If $\delta_{j+p+1}(s) \neq 0$ and $\operatorname{sign} \delta_{j+p}(s) = \operatorname{sign} \delta_{j+p+1}(s)$ then $j+p+2 \in J(s)$, if $\operatorname{sign} \delta_{j+p}(s) \neq \operatorname{sign} \delta_{j+p+1}(s)$ then in order to see if $j+p+2 \in J(s)$ we form the function $s_{j+p+2}$ in accordance with [2, Remark 6.2]. If it has a pole at $z = 0$ of order $q$ then $j+p+2, j+p+3, \ldots, j+p+q \notin J(s)$, but $j+p+q+1 \in J(s)$.

If $j \in \mathbb{N}$ is such that $\delta_{j-1}(s)$ and $\delta_j(s)$ are nonzero and have the same sign, then the functions $s_j$ and $s_{j+1}$ are defined and their Schur kernels have the same number of negative squares. If $\operatorname{sign} \delta_{j-1}(s) \times \operatorname{sign} \delta_j(s) = -1$ and $s_j$ is defined then $s_{j+1}$ can always be defined in accordance with [2, Remark 6.2], but it may not belong to the class $S^0$.

## References

[1] Andersson LE (1990) Algorithms for solving inverse eigenvalue problems for Sturm–Liouville equations. In: Sabatier PC (ed) Inverse Problems in Action. Berlin Heidelberg New York: Springer
[2] Alpay D, Azizov TYa, Dijksma A, Langer H (2001) The Schur algorithm for generalized Schur functions I: Coisometric realizations. Operator Theory: Adv Appl **129**: 1–36
[3] Alpay D, Constantinescu T, Dijksma A, Rovnyak J (2002) Notes on interpolation in the generalized Schur class I. Applications of realization theory. Operator Theory: Adv Appl **134**: 67–97
[4] Alpay D, Dijksma A, Rovnyak J, de Snoo H (1997) Schur Functions, Operator Colligations, and Reproducing Kernel Pontryagin Spaces. Operator Theory: Adv Appl **96**
[5] Bertin MJ, Decomps-Guilloux A, Grandet-Hugot M, Pathiaux-Delfosse M, Schreiber JP (1992) Pisot and Salem Numbers. Basel: Birkhäuser
[6] Bart H, Gohberg I, Kaashoek MA (1979) Minimal Factorization of Matrix and Operator Functions. Operator Theory: Adv Appl **1**
[7] Chamfy C (1958) Fonctions méromorphes sur le circle unité et leurs séries de Taylor. Ann Inst Fourier **8**: 211–251
[8] Ćurgus B, Dijksma A, Langer H, de Snoo HSV (1989) Characteristic functions of unitary colligations and of bounded operators in Krein spaces. Operator Theory: Adv Appl **41**: 125–152
[9] Dufresnoy J (1958) Sur le probléme des coefficients par certaines fonctions dans le cercle unité. Ann Acad Sci Fenn Ser AI **250**(9): 1–7
[10] Delsarte P, Genin Y, Kamp Y (1991) Pseudo-Carathéodory functions and hermitian Toeplitz matrices. Philips J Res **41**: 1–54
[11] Gladwell GML (1991) The application of Schur's algorithm to an inverse eigenvalue problem. Inverse Problems **7**: 557–565
[12] Iokhvidov IS (1982) Hankel and Toeplitz Matrices and Forms, Algebraic Theory. Boston, Mass: Birkhäuser
[13] Krein MG, Langer H (1977) Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume $\Pi_\kappa$ zusammenhängen, Teil I: Einige Funktionenklassen und ihre Darstellungen. Math Nachr **77**: 187–236
[14] Schur I (1917) Über die Potenzreihen, die im Innern des Einheitskreises beschränkt sind. J Reine Angew Math **147**: 205–232; English translation in: I. Schur Methods in Operator Theory and Signal Processing. Operator Theory: Adv Appl **18**: 31–59
[15] Schur I (1918) Über die Potenzreihen, die im Innern des Einheitskreises beschränkt sind; Fortsetzung. J Reine Angew Math **148**: 122–145; English translation in: I. Schur Methods in Operator Theory and Signal Processing. Operator Theory: Adv Appl **18**: 31–59

Authors' addresses: D. Alpay, Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, 84105 Beer Sheva, Israel, e-mail: dany@ivory.bgu.ac.il; T. Ya. Azizov, Department of Mathematics, Voronezh State University, 394693 Voronezh, Russia, e-mail: azizov@tom.vsu.ru; A. Dijksma, Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands, e-mail: dijksma@math.rug.nl; H. Langer, Department of Mathematics, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria, e-mail: hlanger@mail.zserv.tuwien.ac.at